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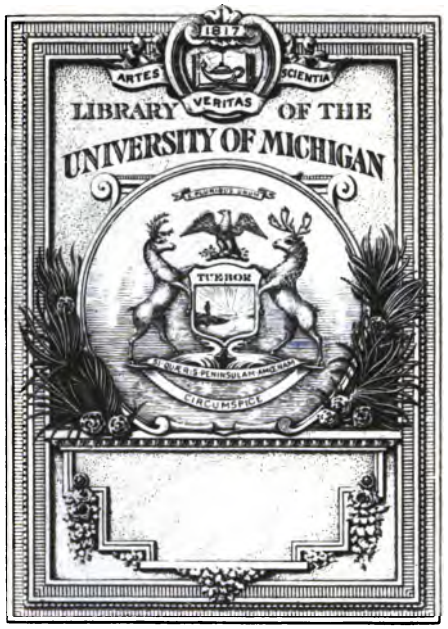
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PRINCIPLES

OF

ALGEBRA.

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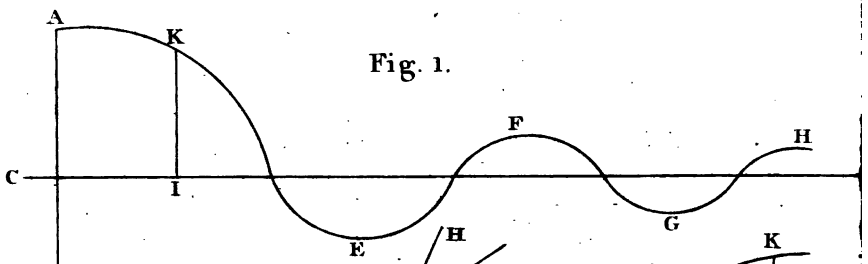

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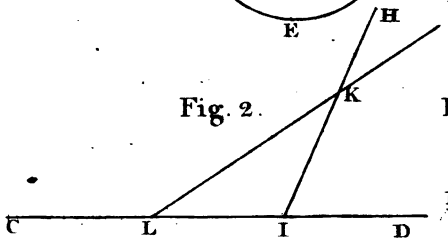

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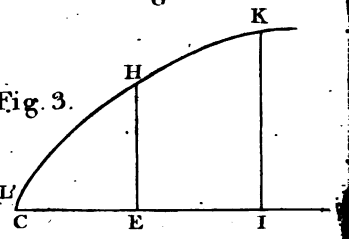

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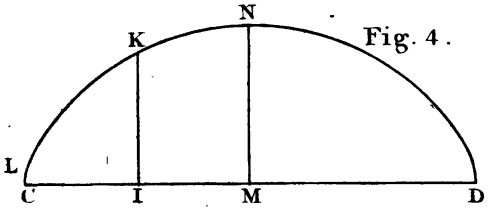

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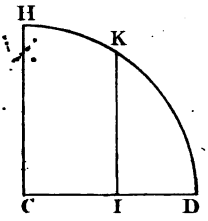

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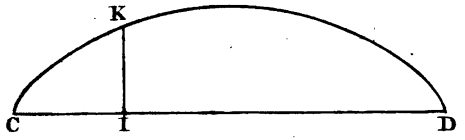

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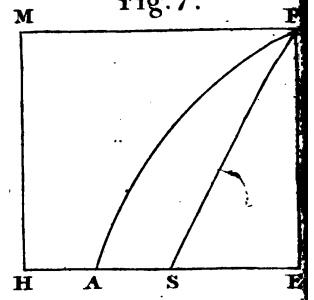

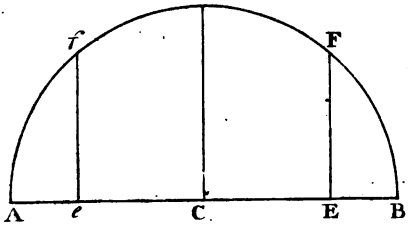

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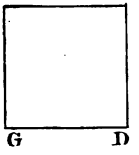

THE

PRINCIPLES

OF

ALGEBRA.

BY

WILLIAM FREND.

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PREFACE.

THE following work is intended for the uſe of ſchools, academies, and colleges; for the latter, on the ſuppoſition that in the former the education of boys has been neglected. On the proper time to put it into the hands of boys and girls, it is impoſſible to determine: but there is nothing in the work which they cannot, by proper inſtruction, comprehend before the age of fifteen. My acquaintance with the maſter of a little day-ſchool gave me an opportunity of making an experiment, which led me to this opinion, and convinced me that the knowledge of numbers is very eaſily acquired, and that it is one of the beſt employments for children. I divided his ſchool into claſſes: the youngeſt hardly knew their letters, the beſt could read only imperfectly, and had made very little progreſs either in writing or accounts. At certain intervals each claſs was queſtioned in numbers; and with the higheſt the beginning was made by aſking them to add two low numbers together, and then in ſucceſſion other two numbers, till it was clearly ſeen how far they could

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go in addition. The ſame was then tried in ſubtraction, and ſo on through multiplication and diviſion. For the firſt day they were awkward, not being accuſtomed to add numbers together without their ſlates; but they ſoon did it with eaſe, and by this mean made a greater progreſs in one month, than they would otherwiſe have done probably by the common mode in the courſe of a year.

Hence I wiſh to recommend to the maſters of ſchools, particularly the great ſchools, ſuch as Eton, the Charterhouſe, Wincheſter, and others approaching to them in ſome degree of reputation for claſſical learning, to adopt a mode by which every boy in their ſchools cannot fail of learning with great eaſe a very uſeful art. I ſhould recommend them to divide their ſchool into claſſes, conſiſting of not more than five boys in each claſs. Each claſs would be diſtinguiſhed by the progreſs which it has made in numbering. The loweſt probably would be able to add only, the higheſt probably to multiply together only two numbers under twenty. Five minutes in the morning and five in the afternoon will be time ſufficient for the exerciſe of each claſs. The head maſter might hear two claſſes. The boys in the firſt claſs might hear thoſe in the third and fourth: The boys in the ſecond claſs, thoſe in the fifth and ſixth claſſes; and thus progreſſively throughout the ſchool. Thus, no boy would be employed more than fifteen minutes on any day, and moſt days only ten minutes,

minutes, in this exercise. By examining the lower classes, the boys in the higher classes would be very much improved: their memory would be strengthened; and in future life they would be as little likely to forget their numbering as they are the shape of the letters in the alphabet. Another advantage proposed by classing boys in this manner is, that they may be in the habit of comparing themselves with each other in more respects than one. An overweening pride is apt at present to get the better of them, from a consciousness of superiority in the Latin or Greek languages; and they are surprised some years after to find, that the boys, whom they despised at school, possess talents which make them the most valuable members in society. Indeed this mode of classing a school would be attended with advantage, if the boys were questioned in geography, history and other things, by which the talents of each might be displayed: thus the vanity of early years would be checked, and men would learn to form a better opinion of themselves and their neighbours.

If upon this suggestion any master of a school should adopt the mode proposed, I should be much obliged to him to acquaint me with the result of his experience; and indeed if the masters, ushers, or tutors of schools, academies or colleges should, on examination, find this work adapted to their use, I should esteem it a favour to have

any faults in it pointed out to me by them, and to receive their hints for future improvement.

To prepare a boy for the reading of this book, we cannot begin too early; and the preparation is ſimple. As ſoon as he begins to write figures, the algebraical marks ſhould be introduced into his copies. Thus his firſt copy might be in addition, after a time in ſubtraction, ſoon after in multiplication, then in diviſion.

$$3 + 4 = 7$$
$$8 - 5 = 3$$
$$9 \times 9 = 81$$
$$24 \div 8 = 3.$$

On ſhewing the copy book, the boy ſhould always read his copy to the maſter. Thus three and four equal ſeven; from eight take five, the remainder equals three; nine into 9 equals eighty-one; twenty-four divided by eight equals three. By degrees a letter may be placed in his copies. Thus $a = 6$. $\therefore 5a = 30$; and thus with very little trouble a boy will by mere reading become not only as well acquainted with the marks $+ - \times \div < = > \therefore$ as with the figures 0 1 2 3 4 5 6 7 8 9, but underſtand the principles laid down for the ſolution of ſimple equations. During this time, the boy, it is preſumed, is learning the firſt rules in arithmetic; and as ſoon as he can add, ſubtract, multiply, and divide whole numbers, and can juſt

do

do a ſum in the rule of three, I recommend that he ſhould enter upon the principles of algebra.

Allowing this mode to be good, ſome one may perhaps aſk me, why I ſhould think of adding to the number of books already written upon this ſubject, and not content myſelf with referring to the authors in common uſe? I am prepared to anſwer the queſtion. Half a dozen years experience, as tutor of a college in the univerſity of Cambridge, taught me the difficulties under which young men labour in endeavouring to learn algebra by the common mode. Some throw away their books before they can do a ſimple equation; others, with more courage, get through equations of the ſecond order, but are afraid to venture on the ſecond part of Maclaurin's Algebra; others wade through a few chapters, but are frightened, and with good reaſon, at Cardan's Rule; the bold ones ruſh forward through thick and thin, till after having made, as they think, ſome curious diſcoveries on the limits of negative roots, they finiſh their courſe in deſpair, in endeavouring to find the number of impoſſible roots in an equation of n dimenſions.

The converſation of theſe learners led me to reflect on the nature of a ſcience which now ſeems to me to be built on the cleareſt principles. The ideas of number are the cleareſt and moſt diſtinct in the human mind; the acts of the mind upon them are equally ſimple and clear.

There

There cannot be confuſion in them, unleſs numbers too great for the comprehenſion of the learner are employed, or ſome arts are uſed which are not juſtifiable. The firſt error in teaching the principles of algebra is obvious on peruſing a few pages only in the firſt part of Maclaurin's Algebra. Numbers are there divided into two ſorts, poſitive and negative; and an attempt is made to explain the nature of negative numbers, by alluſions to book-debts and other arts. Now, when a perſon cannot explain the principles of a ſcience without reference to metaphor, the probability is, that he has never thought accurately upon the ſubject. A number may be greater or leſs than another number; it may be added to, taken from, multiplied into, and divided by another number; but in other reſpects it is very untractable: though the whole world ſhould be deſtroyed, one will be one, and three will be three; and no art whatever can change their nature. You may put a mark before one, which it will obey: it ſubmits to be taken away from another number greater than itſelf, but to attempt to take it away from a number leſs than itſelf is ridiculous. Yet this is attempted by algebraiſts, who talk of a number leſs than nothing, of multiplying a negative number into a negative number and thus producing a poſitive number, of a number being imaginary. Hence they talk of two roots to every equation of the ſecond order, and the learner is to try which will ſucceed in a given equation: they talk of ſolving an equation, which requires two

impoſſible roots to make it ſolvible: they can find out ſome impoſſible numbers, which, being multiplied together, produce unity. This is all jargon, at which common ſenſe recoils; but, from its having been once adopted, like many other figments, it finds the moſt ſtrenuous ſupporters among thoſe who love to take things upon truſt, and hate the labour of a ſerious thought.

Rejecting theſe ſtrange ideas of number, I reſolved to try whether the principles of algebra did not admit of an eaſy explanation, and whether every right ſolution, produced by the common mode from various errors in the premiſſes, and the reaſoning upon them, might not be deduced by a mode of reaſoning, to which there could be no objection. The event has anſwered my expectations, and the reader will ſee the difference between this and the common mode, by comparing the examples in this work with thoſe in other authors. In the ſolution of equations of the ſecond order, he will ſee in what caſes two roots may be diſcovered in an unlimited equation, and, of theſe two roots, which is applicable to his particular queſtion. He will ſee, that there is no difficulty in the change of marks in adding, ſubtracting, multiplying and dividing, as it neceſſarily reſults from the nature of the compound term. He will no longer trouble himſelf with the number of negative roots in an equation, as there cannot be ſuch things in an equation; and if there ſhould be an impoſſible root in the concluſion, he will impute it to the

proper

proper cauſe, either to an error in his mode of reaſoning, or to falſe premiſſes.

Some care has been taken in endeavouring to adapt the language for the perſons to whoſe uſe this book is dedicated, that is, to Engliſh boys and girls. Hence the terms quadratic, cubic, biquadratic, and the like, as applied to equations, are exploded; and the words ſquare, cube, ſolid, ſurſolid, as applied to numbers, are for the ſame reaſon rejected. But habit will ſometimes prevail over our beſt deſigns. Thus, from the long uſe of the word ſquare, it eſcaped my correcting hand in page 105, line 12, where, for ſquare, the words "ſecond power" ſhould be inſerted. Square and cube are modes of continued quantity, and cannot be applied to numbers: the abſurdity is ſeen in the uſe of the word ſurſolid; for, if there could be ſuch a thing as a ſolid number, there might be a ſurſolid number, and a thing might be more than ſolid, which is abſurd. People err much in ſuppoſing that a word is of little conſequence, if it is explained. If that word has a very different meaning in other reſpects, the learner will confound frequently the different meanings, and paſs through life without having a clear idea upon the ſubject. In educating children, we ſhould take care not to uſe a word above their comprehenſion, nor, by our authority, to impreſs a poſition on their minds which is not true. If we teach them little, we ſhould teach them that little well: but we are doing them a real injury,

injury, when we fill their heads with a jumble of words, or with falſe and incoherent notions.

It was not my intention, even if I had been able, to advance the bounds of ſcience by this work. There are ſeveral things alſo omitted, which, if the work had been intended for higher capacities, would have been inſerted. Thus, the proof of the binomial theorem would have been given, and the limits of the roots of equations would have been enlarged upon. I had alſo drawn up a ſeries of problems; but I deferred the publiſhing of them, both that I might ſee the reception which my ideas might meet with from the public, and alſo becauſe I was unwilling that the learner ſhould be prevented, by too long practice on ſome of the rules, from having as ſoon as poſſible a general idea of the principles. After he has learned the mode of ſolving ſimple equations, the examples following it will, with the ſections on fractions and decimal fractions, afford him ſufficient practice. The ſection on powers and roots ſhould not engage much of his attention at firſt: it will be ſufficient to read the 82d, 83d, and 84th pages, and the 91ſt, beginning with the fifth line, 92d, 93d, 94th, 95th, that he may be able to find the ſecond root of any number. Then he may proceed to page 100, to the ſolution of equations of the ſecond order, returning to the ſection on powers, when he comes to the reſolution of an equation of the third order, that he may be able to find the third root of a number. The ſections on the ſumming

ſumming of ſeries will afford him farther practice: and if he makes himſelf maſter of the principles as laid down in this work, he will find, it is preſumed, little difficulty in the future application of them.

Upon the encouragement given to this work by maſters of ſchools depends the execution of the whole of my plan, which was to lay down, in another volume, the principles of fluxions, and the method of increments and differences; to explain farther the higher parts, as they are called, of algebra; and to give a ſeries of problems adapted to the two volumes. Having thus initiated a lad into pure mathematics, I ſhould proceed to the mixed, and I ſhould endeavour to explain the four branches of natural philoſophy, mechanics, hydroſtatics, optics, aſtronomy, in ſeparate works, for the uſe of ſchools, keeping in view my firſt idea, that the greater part of the mathematics now taught in the univerſity of Cambridge may be made level to the capacities of boys and girls under ſeventeen years of age. This notion will not appear very extraordinary, except to thoſe who do not reflect on the progreſs of the human mind. Young men of twenty years of age now read with eaſe the Principia of Sir Iſaac Newton, which, a hundred years ago, were thought too abſtruſe for the greateſt adepts in mathematics.

WILLIAM FREND.

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ERRATA.

Page 13, lines 19, 20, *read* $3c - 4c$, and $4b - 4c$.
18, line 13, *read* $2bc - c^2$.
38, line 4, *read* — as well as x.
61, line 20, *read* $= d \times \overline{mnp + m + 1}$,
62, line 3, *read* $= np + 1$.
95, line 9, *read* $+ q^2$.
98, line 17, *read* $3a^2b$.
104, line 11, *read* x^2, and $2c$.
105, line 12, *read* its ſecond power.
108, line 11, *read* $+ \frac{5}{6} = x$.
145, line 7, *read* $\overline{\frac{2}{x} + 1}$.
159, lines 15, 16, *read* $= x^2 + 9$,
—— line 17, *read* $6x - x^2$.
174, line 1, *read* becauſe r.
183, line 5, *read* $= \sqrt[5]{1024}$.

PRINCIPLES

OF

ALGEBRA.

ALGEBRA is the ſcience which teaches the general properties and relations of numbers. The word is derived from the Arabians, who, from the great utility of the ſcience, gave to it the name of the excellent. In common arithmetick the concluſion drawn from certain poſitions is confined entirely to the poſitions expreſſed by certain numbers: in algebra the concluſion drawn is general, applying to any numbers whatever which may be in the relation determined by the queſtion. Thus, if it is propoſed to find the price of twenty hats, when thirty coſt fifteen pounds, the anſwer refers to that ſingle queſtion only, and is ten pounds. By algebra the queſtion would have been reſolved generally: what is the price of any number of hats, when a known number coſts a known number of pounds? and the anſwer is in general terms applicable to the former, and to all other ſimilar caſes.

Algebra is therefore only the ſcience of numbers improved: and, when it can be applied to other branches of ſcience, the terms uſed always expreſs numbers.

In arithmetick, by means of ten figures, repeated when neceſſary, any number whatever may be expreſſed: but the order, in which they are placed, ſo determines their ſignification, that they cannot in that ſituation expreſs any other numbers. In algebra, certain characters are uſed, by which any number whatever may be expreſſed: but each character may denote a variety of invariable numbers, or they may repreſent any numbers unknown whoſe value is by various modes to be diſcovered.

Algebraical characters are of three kinds: being either marks of numbers, or of the relations of numbers to each other, or of the mode of working by numbers.

The marks of numbers are the common figures in arithmetick, or the letters of the engliſh and foreign alphabets. The letters in the former part of the alphabet denote known numbers: the letters in the latter part of the alphabet denote unknown numbers. Thus a, b, c, d, e, repreſent known numbers, as 3, 5, 27, 39, 666. x, y, z, repreſent unknown numbers. In the queſtion above given, the number of hats would have been repreſented by a and b, the coſt of the former quantity by c, and the coſt of the latter by x: and x is equal to b multiplied into c divided by a.

A number, or ſet of numbers, muſt be either greater than, equal to, or leſs than another number or ſet of numbers,

bers, and theſe relations are diſtinguiſhed by three marks, $< = >$.

$<$ is the mark of greater than; thus $a < b$ denotes, that a is greater than b, or a contains b.

$=$ is the mark of equality; thus $a = b$ denotes, that a is equal to b.

$>$ is the mark of leſs than; thus $a > b$ denotes, that a is leſs than b, or a is contained in b.

Numbers may be added to, taken from, multiplied into, or divided by other numbers: and each ſtep is diſtinguiſhed by its particular mark.

$+$ is the mark of adding: $a + b$ means, that b is to be added to a; and we read it thus, to a add b.

$-$ is the mark of taking away: $a - b$ means, that b is to be taken away from a; and we read it thus, from a take b.

$\times$ is the mark of multiplying: $a \times b$ means, that a is to be multiplied into b; and when two letters come together without any mark between them, this mark is underſtood. Thus ab is the ſame as $a \times b$, and frequently a dot is put between the two numbers to expreſs the ſame thing: thus $a . b$, $13ab . c$, $3 . ab$; and we read ab thus, a into b.

$\div$ is the mark of dividing: $a \div b$ means, that a is to be divided by b; but frequently the numbers themſelves are put into the places of the dots; thus $\frac{a}{b}$ means alſo that a is to be divided by b; and we read it thus, a by b.

$\backsim$ is the mark of difference: $a \backsim b$ means the difference of the two numbers a and b, which will be either $a - b$ or $b - a$, according as a is greater or leſs than b; and we read $a \backsim b$ thus, the difference of a and b.

Thus then we may read with eaſe algebraical writing. For $+$ ſay add; for $-$ take (that is take away); for $\times$ into (that is, multiply into); for $\div$ by (that is, divide by); for $\backsim$ difference of; for $=$ is equal to. Then $a - b + c \times \frac{d}{e} = f \backsim g$ may be read, from a take b, add c into d by e, and it is equal to the difference of f and g.

In writing down numbers with their relations in algebra, it is improper to begin with any of the marks but thoſe of number. The number muſt be firſt written down to or from which another is to be added or taken away, into or by which another is to be multiplied or divided. The ſame is alſo obviouſly neceſſary for the marks of equality or difference. It is improper to write thus: $- a + b \times c = \frac{d}{e}$; it ſhould be $b \times c - a = \frac{d}{e}$. Again, to write thus, $b \times c - a = - f$, is not only improper, but abſurd, as will be ſeen by attempting to read it. From b into c take a, the remainder is equal to ſome number which we will call m; but the mark $-$ before f denotes that it is to be taken away from ſome number which is not written down, and we cannot make any ſenſe of the expreſſion $- f$.

Again, we ſhould do worſe by writing down any of the above marks without any numbers. Thus $- \times - = +$ is

as

as nonſenſical in algebra, as in common language to ſay, take away into take away equals add.

A number ſtanding by itſelf, or a ſet of numbers with the marks of adding or taking away between them, is called a term, and a term is either a ſimple or a compound term.

A ſimple term is either without any or with only one mark prefixed to it. Thus a ſtanding by itſelf is a ſimple term: as alſo are, in any expreſſion, theſe terms $+aa, -abc, +abde, -\frac{ab}{c}, +\frac{12.ad}{g}, +\frac{5x}{z}$.

A compound term conſiſts of ſeveral ſimple terms. As $a+b, a-b, a+b+c, 3ab+\frac{4c}{d}+f$.

A ſimple term may be either ſingle when it conſiſts of a ſingle number as $1, a, 10, m, z$; or it may have ſeparate parts, letters, figures, or both: parts known and unknown. Thus ax is a ſimple term, conſiſting of two literal parts; a the known part, x the unknown part. ab has two literal known parts. 3×5 has two parts, both figures. $13ax$ has three parts, one a figure, the other two letters. Or it may be divided into two parts; $13a$ the known part, and x the unknown part. Of theſe parts it is often uſeful to conſider one as the principal, and the others as co-parts. Thus in the ſimple term $5ax$, if x is conſidered as the principal, $5a$ may be called the co-part: or 5 is the figure co-part, and a the literal co-part.

Simple terms are called like, when they conſiſt of ſimilar parts. Thus $3a$ and $5a$ are like terms, 3000 and 5000,

$7abc$ and $100abc$, $27cbx$ and $27000cbx$. Simple terms are unlike, when the parts of which they are made are different. Thus $2ab$ and $3bc$, aay and cbk, $5bcd$ and $7gck$.

If a compound term conſiſts of two ſimple terms, it may be called a double term; as $a + b$, $4a - 8z$. If it conſiſts of three terms, it may be called a treble term; as $a + b - c$, $3a + bc + 2d$. If it conſiſts of four terms, it may be called a fourfold term, as $ab + cd - ef + g$; and if it conſiſts of a number of terms not neceſſary to be mentioned; it is called a manifold term, as $2ab + 2bc - 2bd + 3ed - 4fg + ghh$.

A line is frequently drawn over a compound term to diſtinguiſh it, which may be called a band, as it binds the terms together; and then each part of the compound term is affected by the mark prefixed to or following the compound term. Thus $\overline{a + b} \times a$ means that both a and b are to be multiplied into a. $a - \overline{a - b + c}$ means that the compound term is to be taken away from a. In dividing numbers this band is omitted: $\frac{a + b + c}{a + b}$ means that the compound term $a + b + c$ is to be divided by the compound term $a + b$: but in the upper and lower parts there may be ſeveral compound terms, and then the band is retained; thus $\frac{\overline{a + b} \times \overline{a - b}}{\overline{a - e} \times f}$ means that the product of the compound terms $\overline{a + b}$ into $\overline{a - b}$ is to be divided by the product of $\overline{a - e}$ into f.

Inſtead of repeating a letter in a term, it is uſual to put a figure over it, denoting the number of times it is to be found in that term. Thus a^2 means the ſame as aa, a^3 the

the ſame as *aaa*, a^2b the ſame as *aab*, and $\overline{a+x}^2$ the ſame as $\overline{a+x} \times \overline{a+x}$. Theſe figures are called indexes, and the terms having theſe indexes are called powers. a^2 is the ſecond power of a, a^3 is the third power of a, a^n is the n^{th} power of a, or a repeated n times. The ſame thing is often found uſeful with the common numbers; thus 3^3 is the ſame as $3 \times 3 \times 3$, 5^4 is the ſame as $5 \times 5 \times 5 \times 5$.

An index placed at the left hand of a term, having a bent line between it and the term, ſhews that the term expreſſes a number, which being repeated in a ſimple term as often as there are units in the index, is equal to the term under the bent line. This number is called the root of the term. Thus $\sqrt[2]{9}$ means a number which multiplied into itſelf is equal to 9. $\sqrt[3]{64}$ means a number which being repeated three times makes 64; and as 3×3 make 9, and $4 \times 4 \times 4$ make 64, the terms 3, or $\sqrt[2]{3 \times 3}$, or $\sqrt[2]{9}$ mean the ſame number, as alſo do, 4, $\sqrt[3]{4 \times 4 \times 4}$, and $\sqrt[3]{64}$. In the ſame manner $\sqrt[2]{m}$ means a number which, repeated twice, is equal to m, $\sqrt[3]{a^2 - b^2}$ means a number which, being repeated thrice, is equal to $a^2 - b^2$. Roots are diſtinguiſhed by their indexes, in the ſame manner as the powers: Thus $\sqrt[2]{a}$ is the ſecond root of a, $\sqrt[3]{a}$ the third root of a, $\sqrt[n]{a}$ the n^{th} root of a. When the bent line is without an index, the index 2 is underſtood.

ADDITION.

SIMPLE terms, which are like, are added together as in common arithmetick. a, $2a$, $3a$, $4a$, added together, make $10a$; as a thouſand, two thouſand, three thouſand, four thouſand added together make ten thouſand.

If the ſimple terms are unlike, they are to be written down together with the mark of addition between them: a, $3b$, $5c$, $8d$, are thus added $a + 3b + 5c + 8d$. In common arithmetick the order of the figures ſerves inſtead of the mark of addition: thus, to add together 1000, 300, and 52, we write down 1352.

To add together compound terms, which have not any term to be taken away from the others, bring together the like terms; then add them together, and to the ſum affix the unlike terms with the mark of addition before them. Thus, to add $a + 3b + 8c$, to $5a + 4b + 12c$, and $3a + 9b + 12c + 7e$, bring the like terms together.

a	$3b$	$8c$
$5a$	$4b$	$12c$
$3a$	$9b$	$12c$
$9a$	$16b$	$32c$

The ſum is $9a + 16b + 32c + 7e$.

Add together $5a + 3x + 7f$, $4a + 2b + 3c$,
$7b + 5c + 8y + 2d$, $13a + 9d + 15f + 7y$.

The ſum is $22a + 22f + 9b + 8c + 11d + 3x + 15y$.

Add

Add together $12b + 7m + 12bc$, $3xy + 4bc + 8n$,
$5bc + 14xy + 18m + z$, $2b + 3bc + 5xy$.

The ſum is $14b + 25m + 24bc + 22xy + 8n + z$.

SUBTRACTION.

A SIMPLE term is taken away from another ſimple term of the ſame kind, in the ſame manner as in common arithmetick. Thus, to take away $4ab$ from $10ab$, place the former term under the latter.

From	$10ab$	From	10 thouſand
Take	$4ab$,	Take	4 thouſand,
Remainder	$6ab$	Remainder	6 thouſand.

From	$18x$	From	$59abc$
Take	$14x$	Take	$32abc$
Remainder	$4x$	Remainder	$27abc$.

It is evident that the term to be taken away muſt be leſs than the term from which it is to be taken.

If the ſimple terms are not like, write them down with the mark of taking away before the term to be taken away. Thus if it is required to take away b from a, it cannot be actually done; but inſtead of doing it, the terms are thus written down, $a - b$; and the extenſive uſe of this mode of writing down characters will be hereafter clearly ſeen.

A compound term, having only the mark of addition between the ſimple terms, is taken away from a ſimilar compound term, by placing like terms under like, and taking away as before. Thus, from $5a + 6b$ take away $a + 2b$.

From	$5a + 6b$	From	$14a + 12cd + 15z$
Take	$a + 2b$	Take	$3a + 11cd + 14z$.
Remainder	$4a + 4b$.	Remaind.	$11a + cd + z$.

From $18a + 7z + 9b$
Take $3a + 2z$

Remainder $15a + 5z + 9b$.

From $7a + 18x + 15cd + 3mn$, take away $5x + 12cd + 4a$. The remainder is $3a + 13x + 3cd + 3mn$.

From $12abc + 49ab + 54a + m + 12y$, take away $m + 44a + 3y$. The remainder is $12abc + 49ab + 10a + 9y$.

From $28k + 43xy + 17c + z + b$, take away $z + 16c + b$. The remainder is $28k + 43xy + c$.

If there are terms in the compound term to be taken away, unlike any terms in the compound term from which they are to be taken, affix them to the reſult with the ſign of taking away before them.

From $5a + 2b$
Take $3a + 7c$

Remainder $2a + 2b - 7c$.

The

The number $2a + 2b$ repreſents the reſult on taking away $3a$; and as we had no means of taking away $7c$, we can aſſert only that the compound term $2a + 2b - 7c$ gives the true reſult, which would be derived by taking away $7c$ from $2a + 2b$.

From $8a + 14y + 12$, take away $8a + 5y + 12 + 6c + d$. Reſult is $9y - 6c - d$.

From $7z + 8m + 42k$, take $3b + 42k + b$. Reſult is $7z + 8m - 3b - b$.

When a compound term to be taken away has any ſimple terms with marks of taking away before them, thoſe terms are to be added to the remainder.

From $3a + b$
Take $a - 2b$

Reſult $2a + b + 2b$.

By taking away a the reſult is $2a + b$, but by ſo doing a greater number is taken away than the queſtion demands; for the difference only between a and $2b$ ſhould have been taken away. The reſult therefore is too ſmall by that quantity $2b$, which muſt be added to give the true reſult $2a + b + 2b$.

Let a be equal to 4 and b to 1, then $3a + b$ make 13, and $a - 2b$ make 2. If we take away 4 from 13, the remainder is nine; but we ſhould not take away a or 4, but $4 - 2$, that is 2, and the true reſult is 11, or $9 + 2$.

From $5a + 7b + 8c$ take away $3b - 4a$. The remainder is $4b + 5a + 8c + 4a$.

From

From $12x + 15y + 36$ take $3x - 7y + 32$. The result is $9x + 15y + 4 + 7y$.

If the compound term from which another is to be taken away contains ſimple terms with the mark of taking away before them, affix theſe terms to the reſult derived from the preceding rule with the ſame ſign before them.

From $3a - 4b$
Take $2a - 2b$

$a - 4b + 2b$.

By taking away $2a - 2b$ from $3a$ we gain the reſult $a + 2b$; but the number from which the other was to be taken was not $3a$ but $3a - 4b$; therefore $4b$ is ſtill to be taken away from the remainder of $3a$, after the firſt taking away, to give the true reſult.

From $5a + 4b - 8z$ take $3a + 8z + 4b - c$. The reſult is $2a - 8z - 8z + c$.

From $7a + b - m$ take $4a + b - e$. Reſult $3a + e - m$.

From $a^2 + 2ab + b^2$ take $a^2 - 2ab + b^2$. The reſult is $2ab + 2ab$.

From $a^3 - 3a^2b + 3ab^2 - b^3$ take $a^2 + 2ab + b^2$. The reſult is $a^3 - 3a^2b + 3ab^2 - b^3 - a^2 - 2ab - b^2$.

By acting in this manner a compound term will frequently reſult in which are like terms with the ſame marks before them, and the compound may be ſhortened by

by adding together the like terms and retaining their marks.

From $9a - 13b + 5c$ take $5a - 9b - 6c$; the result, according to the preceding rules, is $4a - 13b + 5c + 9b + 6c$, in which $5c$ and $6c$ have the mark of adding before them, and may therefore be added together and make $11c$, and the term becomes $4a - 13b + 11c + 9b$.

Now $13b$ and $9b$ are like terms, with unlike marks; and $13b$ cannot be taken away from $9b$; yet, as $13b$ may be considered as made up of two parts, $9b$ and $4b$, we may write our term down thus, $4a - 9b - 4b + 11c + 9b$; and now taking away $9b$ from $9b$, there remains for the true sum $4a - 4b + 11c$.

When therefore there are like terms, having unlike marks in a compound term, separate the greater term into two parts, one of which is equal to the smaller term, and the compound term may be made less; $5a - 8b + 3c + 4b - 7c$, is the same as $5a - 4b - 4b + 3c + 4b - 3c + 4c$, which, by taking away $4b$ from $4b$, and $3c$ from $3c$, becomes $5a - 4b + 4c$; in which it is to be noted, that the part of the greater term remaining has always the mark of that term.

From $a^2 + 2ax + x^2$ take $a^2 - 2ax + x^2$; the result is $4ax$.

From $a^3 + 3ba^2 + 3a^2b^1 + b^3$ take $a^3 - 3ba^2 + 3a^2b^2 - b^3$; the result is $6ba^2 + 2b^3$.

From $5a^2 + 14b^2 - 8c + 12d$ take $3b^2 + 2a^2 - 8c - 5d$; the result is $3a^2 + 11b^2 + 17d$.

Hence

Hence we may learn to add compound terms together which have like terms with unlike marks. For $a + b$ being added to $a - b$, makes the compound term $a + b + a - b$, which is $2a$.

To	$a + 5b$	To	$5a - 7b$
Add	$a - 8b$	Add	$12a + 8b$
Sum is	$2a - 3b$	Sum is	$17a + b$

To	$9a - 6b$	To	$a^2 + 2ax + x^2$
Add	$12a - 18b$	Add	$a^2 - 2ax + x^2$
Sum	$21a - 24b$	Sum is	$2a^2 + 2x^2$

GENERAL RULES

FOR

ADDITION AND SUBTRACTION.

TO add compound terms together, which have like ſimple terms, with like marks, add the like terms together, and affix the common mark.

To	$a + 3b - 2c$	To	$6a - 12b - 9z$
Add	$a + 2b - 5c$	Add	$8a - 7b - 10z$
Sum	$2a + 5b - 7c$	Sum	$14a - 19b - 19z$

If the marks of the like terms are unlike, take the leſs from the greater, and affix the mark of the greater.

To

To $3a - 2b + 3c$	To $10a + 12z - 14x$
Add $5a + 6b - 7c$	Add $9a - 7z + 8x$
Sum $8a + 4b - 4c$	Sum $19a + 5z - 6x$.

For Subtraction, change the marks of the terms to be taken away, and then proceed by the above rules for Addition. From $6a - 5b + 3c$ take away $2a - 7b + 8c$; the remainder is the ſame as the ſum of $6a - 5b + 3c$, $7b - 2a - 8c$.

To $6a - 5b + 3c$	From $12a - 6b + 8c$
*Add $-2a + 7b - 8c$	Take $4a + 5b - 10c$
Sum $4a + 2b - 5c$	Remain. $8a - 11b + 18c$.

From $4a - 8ab + 12ac$	From $a^3 - 3a^2x + b^3$
Take $13ac - a + 10ab$	Take $4c - 3a^3 + b^3$.

It is recommended to a beginner to add and take away compound terms for ſome time by the longer mode, that he may the more eaſily ſee the grounds on which the general rules are formed, and be able to apply them better to practice. From not conſidering theſe things with care at firſt, learners are apt to confound together Addition and Subtraction, and to pervert the uſe of the marks for each.

* We may ſeem here to tranſgreſs our rule of beginning always with a number; but the fact is, that the under line is only a continuation of the upper line, like terms being put under like terms for convenience; and the line is to be read thus: add $7b$, and take away $2a$ and $8c$.

MULTI-

MULTIPLICATION.

IF the simple terms to be multiplied consist only of letters, the letters are to be written down together in one term, which implies that Multiplication. Thus, if it is required to multiply a into a, the term aa is made, which means that a is to be multiplied into a. If ab is to be multiplied into cd, write down $abcd$.

If there are figures in the simple terms, multiply the figures together, as in common arithmetick, and affix the letters as before. Multiply $3a$ into $5a$, the product is $3.5.aa$; but 3×5 makes 15, and the product is $15.aa$.

Mult.	$6ab$	Mult.	$12.acx$	Mult.	$8xyz$
into	$5bc$	into	$11.abx$	into	$12xyz$
Prod.	$30ab^2c$	Prod.	$132.a^2bcx^2$	Prod.	$96.x^2y^2z^2$

Compound terms, having no mark of taking away between the simple terms, are multiplied together by multiplying each simple term of the one into all the simple terms of the other, and adding together the product of each multiplication.

Mult.	$a + b$
into	$a + b$
a's prod.	$a^2 + ab$
b's prod.	$+ ab + b^2$
Product	$a^2 + 2ab + b^2$

$a^2 + ab$ is the product of $a \times \overline{a + b}$, and $ab + b^2$ is the

the product of $b \times \overline{a+b}$; the sum therefore of these two numbers is the product of $\overline{a+b} \times \overline{a+b}$.

Multiply $5a + 3bc$
into $7a + 6bc$

Product is $35a^2 + 51abc + 18b^2c^2$.

Mult. $a + b + c$
into $a + b + c$

a's prod. $a^2 + ab + ac$
b's prod. $+ ab + b^2 + bc$
c's prod. $+ ac + bc + c^2$

Prod. $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.

In the two laſt lines but one ſpaces are left between ab and b^2, ac and bc, that like terms may, for the eaſe of adding, be kept under like terms.

If the compound term to be multiplied has any terms to be taken away, theſe terms, increaſed by multiplication, will remain in the product to be taken away.

Mult. $a - b$
into a

Prod. $a^2 - ab$.

In the compound term $a - b$, b was to be taken away, and the product, which ariſes from multiplying $a \times a$, increaſes only the term b, but cannot alter the nature of its mark.

Mult. $a - b$
into $a + 2b$

a's prod. $a^2 - ab$
$2b$'s prod. $+ 2ab - 2b^2$

Prod. $a^2 + ab - 2b^2$.

What was ſaid before in multiplying $a \times a - b$ is equally true here, in multiplying $2b \times a - b$, the term ariſing from multiplying $b \times 2b$ muſt remain in the product to be taken away, as the term was to be taken away in the compound term $a - b$.

Mult. $a - b - c$
into $a + b + c$

Prod. $a^2 - b^2 - 2bc - c$

Mult. $5a - 4ab + 3c$
into $3a + 2c + 4ab$

Prod. $15a^2 + 8a^2b + 19ac + 4abc - 16a^2b^2 + 6c^2$.

If in the multiplier there is a term to be taken away, the product of that term, multiplied into the compound term, is to be taken away from the product ariſing from the multiplication of the other terms.

Mult. $a + b$
into $a - b$

from a's prod. $a^2 + ab$
take b's prod. $ab + b^2$

Prod. $a^2 - b^2$.

$a^2 + ab$

$a^2 + ab$ is the result of multiplying $a + b$ into a; but this is too much, for we were to multiply $a + b$ into a number less than a by the number b: we are therefore to take away b times $a + b$ for the true product.

Mult. $8 + 4$
into $8 - 4$

from 8's prod. $64 + 32$
take 4's prod. $32 + 16$

Prod. $64 - 16 = 48 = 12 \times 4$.

Mult. $a + b - c$
into $a - b$

from a's prod. $a^2 + ab - ac$
take b's prod. $ab + bb - bc$

Prod. $a^2 - ac - b^2 + bc$.

Mult. $a - b$
into $a - b$

from a's prod. $a^2 - ab$
take b's prod. $ab - b^2$

Prod. $a^2 - 2ab + b^2$.

Mult. $2a + 3b - 4c$
into $a - b - c$

from a's prod. $2a^2 + 3ab - 4ac$
take b's prod. $2ab + 3b^2 - 4bc$

from $a-b$'s prod. $2a^2 + ab - 4ac - 3b^2 + 4bc$
take c's prod. $2ac + 3bc - 4c^2$

Whole prod. $2a^2 + ab - 6ac - 3b^2 + bc + 4c^2$.

Mult. $12x - 7y$
into $8x - 5y$

Prod. $96x^2 - 56xy - 60xz + 35yz$.

Mult. $4a + 5b - 6c$
into $2a - 3b + 4c$

Prod. $8a^2 - 2ab + 4ac - 15b^2 + 38bc - 24c^2$.

To ſhorten theſe ſteps, the product of each ſeparate multiplication may be written down under the firſt product, by changing the mark of each term in the compound taken away, and then proceeding by addition.

Mult. $a - b$
into $a - b$

a's prod. $a^2 - ab$
b's prod. m changed $- ab + b^2$

Prod. $a^2 - 2ab + b^2$.

This appears from (page 15), where it was ſhewn, that in the taking away of compound terms, the remainder was the ſame, as the ſum derived from adding the compound term with its marks changed to the term from which it was to have been taken away.

Mult. $2a + 3b - 4c$
into $a - b - c$

a's prod. $2a^2 + 3ab - 4ac$
b's prod. with m changed $- 2ab - 3b^2 + 4bc$
c's prod. with m changed $- 2ac - 3bc + 4c^2$

Whole prod. $2a^2 + ab - 6ac - 3b^2 + bc + 4c^2$.

Mult.

Mult. $a + b + c$
into $a - b - c$

a's prod. $a^2 + ab + ac$
b's prod. with m changed $- ab - b^2 - bc$
c's prod. with m changed $- ac - bc - c^2$

Whole prod. $a^2 - b^2 - 2bc - c^2$.

DIVISION.

IF a ſimple literal term is to be divided by another ſimple term with different letters, write down the divider under the term to be divided, and draw a line between them. Let it be required to divide ab by cd. Write down $\frac{ab}{cd}$. This follows from the explanation of the mark for diviſion; for the new term ſhews that ab is to be divided by cd, which cannot be done in that ſtate of the two terms.

If the ſimple term to be divided has any letter in common with the divider, ſtrike out this letter in the upper and lower parts of the new term. Divide abe by acd. Firſt, write down the new term as in the above rule $\frac{abe}{acd}$, ſtrike out a in both parts, and the new term becomes $\frac{be}{cd}$.

For every number divided by itſelf is equal to one. $\frac{a}{a}$ is equal to one, $\frac{ab}{ab}$ is equal to one, $\frac{abe}{acd}$ is equal to $\frac{be}{cd}$.

If there are figures in the upper and lower parts of the new term, and the upper figures can be divided by the lower, the new term may be made ſhorter by this diviſion. Divide $20abc$ by $5bd$. 1ſt, Write down $\frac{20abc}{5bd}$. 2d, Strike out the common letter b, and it becomes $\frac{20ac}{5d}$. Laſt, Divide by 5, and the new term becomes $\frac{4ac}{d}$.

$$64amn \div 16amn = 4.$$

$$12b^2cd \div 3cf = \frac{4b^2d}{f}.$$

$$13xyz \div 5pxy = \frac{13z}{5p}.$$

If a compound term is to be divided by a ſimple term, divide each term in the compound by the ſimple term, and in the quotient retain the marks of the compound term.

Divide $3ab + 2ab$ by ab

Quotient is $\frac{3ab}{ab} + \frac{2ab}{ab}$, which makes $3 + 2$.

Divide $8ab + 6bc + 10ad$ by $2ab$

Quotient is $\frac{8ab}{2ab} + \frac{6bc}{2ab} + \frac{10ad}{2ab}$ that is $4 + \frac{3c}{a} + \frac{5d}{b}$.

Divide

Divide $8bd - 3bc - mn$ by $4bc$

Quotient $\frac{2d}{c} - \frac{3}{4} - \frac{mn}{4.bc}$.

Divide $a^2 + 2bx + x^2$ by a^2

Quotient $1 + \frac{2bx}{a^2} + \frac{x^2}{a^2}$.

If the divider is a compound term, divide the firſt term in the other compound by the firſt term in the divider, and multiply the divider by the quotient, which ſubtract as in common diviſion, and continue by bringing down terms, as figures in common diviſion are brought down.

$$\begin{array}{l} a + b \,)\, aa + 2ab + bb \,(\, a + b \\ \quad aa + ab \\ \quad \hline \quad ab + bb \\ \quad ab + bb \\ \quad \hline \quad 0 + 0. \end{array}$$

Divide $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.

$$\begin{array}{l} a^2 + 2ab + b^2 \,)\, a^3 + 3a^2b + 3ab^2 + b^3 \,(\, a + b \\ \quad a^3 + 2a^2b + ab^2 \\ \quad \hline \quad a^2b + 2ab^2 + b^3 \\ \quad a^2b + 2ab^2 + b^3 \\ \quad \hline \quad 0 + 0 + 0. \end{array}$$

If the compound term to be divided has any terms to be taken away from the others, the quotient will alſo have a term or terms to be taken away, and the mark of taking away will be in the quotient before each term reſulting

from the diviſion of a remainder, which has the mark of taking away before its firſt term.

$$\begin{array}{r|l} a+b & a^2-b^2 \;(\; a-b \\ & a^2+ab \\ \hline & -ab-b^2 \\ & \;\;ab+b^2 \\ \hline & \;\;0+0 \end{array}$$

It appears that the firſt product a^2+ab is greater than the term to be divided, and therefore the quotient a is too great, and ſome number muſt be found, by taking which, from a, the quotient will be right. Now, beſides the remainder ab to be taken away, there is alſo in the term to be divided a term to be taken away, namely, b^2; therefore, by adding it to ab, the compound term $ab+b^2$ is to be divided by $a+b$, and the quotient is to be taken from the preceding quotient: b is to be taken from a, and the true quotient becomes $a-b$. After taking away the ſecond product, there is no remainder, and the diviſion may be proved by multiplying $a+b$ into $a-b$, whoſe product is a^2-b^2.

Divide $a^4-2a^2b^2+b^4$ by $a^2+2ab+b^2$.

$$\begin{array}{r|l} a^2+2ab+b^2 & a^4-2a^2b^2+b^4 \;(\; a^2 \\ & a^4+2a^3b+a^2b^2 \\ \hline & \;\;-2a^3b-3a^2b^2. \end{array}$$

After taking away the firſt product, it appears that there remains to be divided a compound term, conſiſting of two parts, one of which is b^4, and the other $2a^3b+3a^2b^2$; the

latter being to be taken away, the quotient derived from the diviſion of this latter part muſt be taken away from the quotient already found.

$$\begin{array}{r} a^2 + 2ab + b^2 \,)\, 2a^3b + 3a^2b^2 \,(\, 2ab \\ 2a^3b + 4a^2b^2 + 2ab^3 \\ \hline -a^2b^2 - 2ab^3. \end{array}$$

The term $2ab$ is to be annexed to the quotient already found, and $a^2 - 2ab$ is the quotient derived from dividing $a^4 - 2a^2b^2$ by $a^2 + 2ab + b^2$, with a remainder, to be taken away, $2ab^3 + a^2b^2$; if this number then had been added to $a^4 - 2a^2b^2$, there would not have been a remainder.

$$\begin{array}{rl} & 2a^3b + 3a^2b^2 + 2ab^3 + a^2b^2 \\ \text{Take away prod.} & 2a^3b + 4a^2b^2 + 2ab^3 \\ \hline & 0 + 0 \quad + 0. \end{array}$$

Some number then muſt be added to the quotient to make the diviſion of $a^4 - 2a^2b^2$ complete, and this is to be found by dividing $a^2b^2 + 2ab^3$ by $a^2 + 2ab + b^2$.

$$\begin{array}{r} a^2 + 2ab + b^2 \,)\, a^2b^2 + 2ab^3 \,(\, b^2 \\ a^2b^2 + 2ab^3 + b^4 \\ \hline -b^4. \end{array}$$

b^2 is the neareſt number to it, but it leaves a remainder to be taken away b^4; and now $a^2 - 2ab + b^2$ is the quotient reſulting from dividing $a^4 - 2a^2b^2$ by $a^2 + 2ab + b^2$, leaving a remainder to be taken away b^4; but the whole term was $a^4 - 2a^2b^2 + b^4$; therefore b^4 remained to be divided, from which its equal may be taken, and thus there is no remainder, and $a^2 - 2ab + b^2$ is the true quotient.

Inſtead

Inſtead of making a freſh diviſion for every remainder, time may be ſaved by retaining the marks of each remainder; and when the quotient of its firſt term is to be taken away, let the marks of each term of the product of this quotient into the divider be changed.

$$a^2 + 2ab + b^2 \,)\, a^4 - 2a^2b^2 + b^4 \,)\, a^2 - 2ab + b^2$$

$$a^4 + 2a^3b + a^2b^2$$

$$-2a^3b - 3a^2b^2 + b^4$$

$$-2a^3b - 4a^2b^2 - 2ab^3$$

$$a^2b^2 + 2ab^3 + b^4$$

$$a^2b^2 + 2ab^3 + b^4$$

$$0 + 0 + 0.$$

According to the rule for ſubtraction (page 15), the reſult $a^2b^2 + 2ab^3$ is derived by adding the preceding compound term, with its ſigns changed to the compound term above it, which was done before by two ſteps, by firſt taking away $2a^3b + 4a^2b^2 + 2ab^3$ from the term above it, and then by adding the remainder to make a new term to be divided.

Divide $1 + b$ by $1 + a$.

$$1 + a \,)\, 1 + b \,(\, 1 - a + a^2 - a^3$$

$$1 + a$$

$$-a$$

$$-a - a^2$$

$$a^2$$

$$a^2 + a^3$$

$$-a^3$$

$$-a^3 - a^4$$

$$a^4$$

In

In this laſt inſtance the firſt term of the quotient is too great, and the mark of firſt remainder a denotes that the next term is to be taken away; the quotient a makes too great a product, to balance which a^2 muſt be added to the quotient; but by doing this too much is added, and the next term muſt be taken away, and ſo on for ever: the diviſion may be continued without end, and after every taking away there is a remainder.

In its preſent ſtate the quotient is $1 - a + a^2 - a^3$, with remainder $\frac{a^4 + b}{1 + a}$.

Divide $a^6 - 2a^5b - a^4b^2 + 4a^3b^3 - a^2b^4 - 2ab5 + b^6$ by $a^2 + 2ab + b^2$.

$$
\begin{array}{l}
a^2+2ab+b^2)a^6-2a5b-a4b^2+4a^3b^3-a^2b^4-2ab5+b^6(a^4-4a^3b \\
\quad a^6+2a^5b+a4b^2 \qquad\qquad +6a^2b^2-4ab^3+b4 \\
\quad \overline{\qquad\qquad} \\
\quad -4a5b-2a^4b^2+4a^3b^3 \\
\quad -4a^5b-8a^4b^2-4a^3b^3 \\
\quad \overline{\qquad\qquad\qquad} \\
\quad\quad 6a^4b^2+8a^3b^3-a^2b^4 \\
\quad\quad 6a^4b^2+12a^3b^3+6a^2b4 \\
\quad\quad \overline{\qquad\qquad\qquad} \\
\quad\quad\quad -4a^3b^3-7a^2b^4-2ab5 \\
\quad\quad\quad -4a^3b^3-8a^2b^4-4ab^5 \\
\quad\quad\quad \overline{\qquad\qquad\qquad} \\
\quad\quad\quad\quad a^2b^4+2ab^5+b^6 \\
\quad\quad\quad\quad a^2b^4+2ab^5+b^6 \\
\quad\quad\quad\quad \overline{\qquad\qquad} \\
\quad\quad\quad\quad 0+0+0.
\end{array}
$$

In this inſtance each term is taken down from the term to be divided, as it is wanted; and in doing this the learner is to obſerve, whether there is a term in any product to be taken

taken away, like any term in the number to be divided; and if there is, he brings down that term, and affixes it to the line above the product.

If there is a term to be taken away in the divider, the rule is not altered, and the diviſion is equally ſimple.

Divide $a^2 - x^2$ by $a + x$.

$$
\begin{array}{l}
a - x \,)\, a^2 - x^2 \,(\, a + x \\
\quad a^2 - ax \\
\quad \overline{\qquad\qquad} \\
\qquad\quad ax - x^2 \\
\qquad\quad ax - x^2 \\
\qquad\quad \overline{\qquad\qquad} \\
\qquad\quad 0 - 0.
\end{array}
$$

Divide 1 by $1 - x$.

$$
\begin{array}{l}
1 - x \,)\, 1 \quad (\, 1 + x + x^2 \;\&c. \\
\quad 1 - x \\
\quad \overline{\qquad\qquad} \\
\qquad\quad x \\
\qquad\quad x - x^2 \\
\qquad\quad \overline{\qquad\qquad} \\
\qquad\qquad\quad x^2 \\
\qquad\qquad\quad x^2 - x^3 \\
\qquad\qquad\quad \overline{\qquad\qquad} \\
\qquad\qquad\qquad\quad x^3.
\end{array}
$$

EQUA-

EQUATIONS.

AN equation is a ſentence in algebraical characters, expreſſing the equality of two terms. The propoſition that $8 + 3$ is equal to 11, is an equation, and is thus written down, $8 + 3 = 11$. The following are alſo inſtances of equations.

$$a = b \qquad a = b + c$$
$$a + 2x - 3c = 4d + e + f$$
$$mn + ab - \frac{p}{q} = \frac{e}{g} + \frac{f}{b}.$$

One great uſe of equations is, by means of known numbers, to find out ſome unknown number in an equation. Thus in the equation $a + 3b - 5x = 4x + d - e$; all the numbers are known except x, and this number, by certain rules, may be diſcovered.

Equations are divided into orders, according to the nature of the unknown numbers in them. If the unknown number has not, in any ſimple term, an unknown co-part, and, if found in the upper part of a ſimple term, is nowhere elſe to be found in the lower part of a ſimple term, the equation is a ſimple equation. $a - \frac{4x}{d} = ex + b$ is a ſimple equation; but $a - \frac{4x}{q} = b - \frac{c}{x}$ is not a ſimple equation, becauſe x, in the term $\frac{4x}{d}$, is in the upper part, and, in $\frac{c}{x}$, in the lower part of a ſimple term; and in this caſe the unknown number cannot be diſcovered by the rules common to ſimple equations.

RESO-

RESOLUTION OF EQUATIONS.

To reſolve an equation, is to find the amount of any number in it, in terms of the other numbers; and there are four general undeniable principles on which the reſolution depends.

1. If equal numbers are added to equal numbers, the ſums are equals.

$$a + d = b + c.$$

Add e to one ſide, and f its equal to the other ſide of the equation.

$$a + d + e = b + c + e$$
$$3 + 5 = 8.$$

Add 7 to one ſide, and $3 + 4$, its equal, to the other.

$$3 + 5 + 7 = 8 + 3 + 4.$$

2. If equal numbers are taken from equal numbers, the remainders are equal.

$$a + b = c + d.$$

Take e from one ſide, and f, its equal, from the other.

$$a + b - e = c + d - f$$
$$6 + 4 = 10.$$

Take

Take 7 from one ſide, and 5 + 2 from the other.

$$6 + 4 - 7 = 10 - 5 - 2.$$

3. Equal numbers multiplied into equal numbers give equal products.

$$a = b.$$

Multiply one ſide by c, and the other by d equal to c.

$$\text{then } ac = bd$$
$$5 + 4 = 9.$$

Multiply $\overline{5 + 4}$ into 6, and 9 into 4 + 2.

$$30 + 24 = 36 + 18.$$

4. Equal numbers divided by equal numbers give equal quotients.

$$a = b.$$

Divide a by c, and b by d equal to c.

$$\frac{a}{c} = \frac{b}{d}.$$

$$12 + 36 = 48.$$

Divide 12 + 36 by 6, and 48 by 4 + 2.

$$\frac{12}{6} + \frac{36}{6} = \frac{48}{4 + 2}.$$

$$2 + 6 = 8.$$

On theſe principles are made rules for freeing the unknown number from all the other numbers, ſo that the unknown number ſhall be on one ſide, and the known numbers on the other ſide of the equation.

1. Let $a = x - b$.

Add b to both ſides.

$$a + b = x - b + b$$
$$* \therefore x = a + b.$$

Let $x - 4 = 12$.

Add 4 to both ſides.

$$x - 4 + 4 = 12 + 4$$
$$\therefore x = 16.$$

2. Let $a = x + b$.

Take away b from both ſides.

$$a - b = x + b - b$$
$$\therefore a - b = x.$$

Let $12 + x = 20$.

Take 12 from both ſides.

$$12 + x - 12 = 20 - 12.$$
$$\therefore x = 8.$$

Hence, if the unknown number has any unknown terms to be added to or taken from it, or is itſelf to be added to or taken from any known numbers, the known numbers may be ſtruck out and placed on the other ſide of the equation, with their marks changed, or the unknown number itſelf may be ſtruck out, and placed on the other ſide of the equation, with its mark changed.

* $\therefore$ is the mark for *therefore*.

$a + x$

$$a + x - b = c + 2x - d$$
$$\therefore a - b - c + d = 2x - x = x.$$

2. Let $a + x = b$.

Take a from both ſides.

$$a + x - a = b - a$$
$$\therefore x = b - a.$$

3. Let $\frac{x}{a} = b$.

Multiply both ſides into a.

$$\therefore \frac{ax}{a} = ab.$$
$$\therefore x = ab.$$

Let $\frac{x}{5} = 20$

$$\therefore \frac{5x}{5} = 5 \times 20$$
$$\therefore x = 100.$$

4. Let $ax = b$.

Divide both ſides by a.

$$\therefore \frac{ax}{a} = \frac{b}{a}$$
$$\therefore x = \frac{b}{a}.$$

Let $20x = 100$.

$$\therefore \frac{20x}{20} = \frac{100}{20}$$
$$\therefore x = 5$$

In the inſtance above, $cx - dax$, is a compound term conſiſting of an unknown number, in two ſimple terms, which is made eaſier for practice, by ſeparating it into its two parts, the known and the unknown, x and $c - da$: then putting the band over $c - da$, and inſerting the mark of multiplication between it and x, we have a term $c - da$, by which both ſides is to be divided, and x remains by itſelf on one ſide of the equation.

The carrying of a term over from one ſide of an equation to the other, and changing its mark, is called tranſpoſition: and it is uſual to ſay, Tranſpoſe ſuch a term; by which is meant, Strike it out on one ſide and place it on the other ſide of the equation, with its mark changed.

$$a - x = b.$$

Tranſpoſe x and b, and the equation becomes

$$a - b = x.$$

EQUATIONS, with their ANSWERS, which the Learner is requeſted to ſolve before he proceeds any farther than to read the annexed note.

$2x + a = x + b.$	Anſ. $x = b - a$
$3x + 12 = 27.$	Anſ. $x = 5.$
$5x + 50 = 4x + 56.$	Anſ. $x = 6.$
$ax + 2ab = 3c^2.$	Anſ. $x = \frac{3c^2}{a} - 2b.$
$\frac{x}{5} + 4 = 10.$	Anſ. $x = 30.$

$\frac{x}{2}$ $\frac{4x}{3}$

$\frac{4x}{3} + 24 = 2x + 6.$ Anſ. $x = 27.$

$\frac{x}{5} + \frac{x}{3} = x - 7.$ Anſ. $x = 15.$

$5x - 16 = 3x + 12.$ Anſ. $x = 14.$

$20 - 3y + 8 = 60 - 7y.$ Anſ. $y = 8.$

$ax - b = cx - d.$ Anſ. $x = \frac{d - b}{c - a}.$

$\frac{y}{3} + \frac{y}{4} = 21.$ Anſ. $y = 36.$

$\frac{y+1}{2} + \frac{y+2}{3} = 16 - \frac{y+3}{4}.$ Anſ. $y = 13.$

$a - \frac{b^2}{x} = c.$ Anſ. $x = \frac{b^2}{a - c}.$

$\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d.$ Anſ. $x = \frac{abcd}{bc + ac + ab}.$

$3x - 5 = 23 - x.$ Anſ. $x = 7.$

$x + \frac{x}{2} - \frac{x}{3} = 4x - 17.$ Anſ. $x = 6.$

$\frac{1}{a} + \frac{b}{x} = c.$ Anſ. $x = \frac{ab}{ac - 1}.$

$5 - \frac{x + 4}{11} = x - 3.$ Anſ. $x = 7.$

$x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}.$ Anſ. $x = 5.$

Note! In ſolving theſe equations, the learner will probably, through want of care, ſometimes bring out a wrong ſolution, and this may ariſe from not ſufficiently attending to the marks: Thus, $\frac{x + 4}{11}$ in one equation is to be

taken away from 5, and the term is thus written down $5 - \frac{x+4}{11}$. In multiplying both ſides of the equation by 11, and writing down the product, the learner may forget, that 4 as well as a was to be taken away, and leaving the mark of addition before 4, will make a falſe equation, writing down his term thus: $55 - x + 4$, inſtead of $55 - x - 4$; the mark of ſubtraction before $\frac{x+4}{11}$ denoted, that $\frac{x}{11}$ and $\frac{4}{11}$ were both to be taken away.

Again, in another equation $\frac{2x-4}{3}$ was to be taken away from 12, and in multiplying both ſides of the equation by 3, this becomes $36 - 2x + 4$: but the learner, by forgetting to change the mark before 4, may produce a falſe equation, making this term $36 - 2x - 4$; for $2x - 4$ was to be taken away from 36; and the rule for ſubtraction has ſhewn, that this is to be done by changing the mark for each term.

From 36
take $2x - 4$
Remains $36 - 2x + 4$.

Again, where there is a band over any number of ſimple terms, the learner may ſometimes forget to multiply every ſimple term into the number, into which the whole was to be multiplied.

Again, he may forget to change his marks, and the miſtake will not appear to him in the concluſion: thus, in the equation $ax - b = cx - d$, ſuppoſe him to write down

$ax -$

$ax - cx = b + d$, he will produce an anſwer $x = \frac{b+d}{a-c}$, which, if the true anſwer were not ſet down, he might not immediately diſcover to be falſe, from changing the mark of d improperly in writing down his ſecond equation.

The learner muſt alſo be careful that none of the equations which he produces from the propoſed equation contain an abſurdity: thus, in the firſt equation, by tranſpoſing $2x$ and b, a new equation is produced. $a - b = x - 2x$, which is abſurd; for $2x$ is greater than x, and cannot therefore be taken from it.—The true equation is $2x - x = b - a$. Wherever, on one ſide of an equation, a number is to be taken from one leſs than itſelf, the error is either in the perſon who propoſed, or in him who attempted to ſolve the propoſed equation.

QUESTIONS RESOLVED BY SIMPLE EQUATIONS.

The propoſed queſtion muſt be firſt written down in algebraical characters, which is done by writing x for the unknown number, and other letters or numbers for the known numbers, and prefixing to each term the mark preſcribed by the conditions of the queſtion. Suppoſe an unknown number added to a known is equal to a known number, then in algebraical writing $a + x = b$.

Ex. What number is that, which, being multiplied into four, ſhall produce a number equal to three times the unknown number added to four?

Let x = the unknown number; then $4x$ = the product of it multiplied into 4.

$\overline{4 + x}$ is the unknown number added to 4; and $\overline{4 + x} \times 3$, is the product of three times the unknown number added to 4.

$$\therefore 4x = \overline{4 + x} \times 3 = 12 + 3x$$
$$\therefore 4x - 3x = 12.$$
$$\therefore x = 12.$$
$$\therefore 4x = 48, \text{ and } 48 = \overline{4 + 12} \cdot 3.$$

Divide the number 60 into three ſuch parts, that the firſt may exceed the ſecond by 8, and the third by 16.

$$\text{1ſt number} = x + 16$$
$$\text{2d number} = x + 16 - 8 = x + 8$$
$$\text{3d number} = x.$$

$$\therefore x + x + 16 + x + 8 = 60$$
$$\therefore 3x = 60 - 16 - 8 = 36$$
$$\therefore x = 12.$$

$$\therefore \text{1ſt number} = 12 + 16 = 28$$
$$\text{2d number} = 12 + 8 = 20$$
$$\text{3d number} = 12 \qquad = 12$$
$$\text{Sum} \quad = 60.$$

A labourer

A labourer was hired, on condition, that for every day he worked he ſhould receive 12d, and for every idle day he ſhould forfeit 8d; at the end of 390 days he found that he had paid for his lazineſs what he gained by his labour. How many days did he work, and on how many was he idle?

Let x = number of working days,
then $390 - x$ = number of idle days,
$12x$ = pence received for work,
$\overline{390-x} \times 8$ = pence paid for idleneſs;

$$\therefore 12x = \overline{390-x} \times 8 = 3120 - 8x$$
$$\therefore 12x + 8x = 3120 = 20x$$
$$\therefore x = \frac{3120}{20} = 156$$
$$\therefore 390 - x = 390 - 156 = 234.$$

He worked 156 days, and played 234.

$$156 \times 12^d = 1872^d = 234 \times 8^d.$$

Two travellers, A and B, 360 miles diſtant from each other, ſet out to meet each other. A travels at the rate of 10 miles an hour, B at the rate of 8 miles an hour. How many miles does each travel, and how long are they upon the road?

x = miles A travels over,
$\therefore 360 - x$ = miles which B travels over.

$$\left.\begin{matrix} \frac{x}{10} = \\ \frac{360-x}{8} = \end{matrix}\right\} \text{hours the two travellers were upon the road.}$$

$\therefore \frac{x}{10}$

$$\therefore \frac{x}{10} = \frac{360 - x}{8}$$
$$\therefore 8x = 3600 - 10x$$
$$\therefore 18x = 3600$$
$$\therefore x = 200$$
$$\therefore 360 - x = 160.$$

∴ A travelled 200 miles, B 160, and they were twenty hours upon the road. $\frac{200}{10} = \frac{160}{8} = 20$.

Four perſons make a common purſe: A puts in twice as much as B, B twice as much as C, C twice as much as D. What part of the whole ſtock did each contribute?

Let a = whole ſtock,
x = D's part,
$2x$ = C's part,
$4x$ = B's part,
$8x$ = A's part.

$$\therefore 8x + 4x + 2x + x = a = 15x.$$

$$\therefore x = \frac{a}{15} = \text{D's part},$$
$$2x = \frac{2a}{15} = \text{C's part},$$
$$4x = \frac{4a}{15} = \text{B's part},$$
$$8x = \frac{8a}{15} = \text{A's part}.$$

Suppoſe the common purſe to contain 60 pounds: D's ſhare was 4*l*. C's 8*l*. B's 16*l*. A's 32*l*.

Divide

Divide a given number into two ſuch parts, that one part may contain the other a given number of times.

Let a be the given number,

x = one part,

$\therefore a - x$ is the other part,

and let n = the number of times that one part is to contain the other.

$$\therefore a - x = nx$$

$$\therefore a = nx + x = x \times \overline{n + 1}$$

$$\therefore x = \frac{a}{n + 1}$$

$$a - x = a - \frac{a}{n + 1}.$$

Let $a = 50$, and $n = 24$

then $x = \frac{a}{n+1} = \frac{50}{25} = 2$

$\therefore a - x = 50 - 2 = 48$

and $48 = 2 \times 24$.

Two liquors are mixed together, the one at m ſhillings the gallon, the other at n ſhillings the gallon: Required the value of a mixture made from a given number of gallons of each.

Let a = the number of gallons at m ſhillings,

b = number of gallons at n ſhillings:

$\therefore ma$ = price of firſt number of gallons,

nb = price of 2d number of gallons:

$\therefore ma + nb$ = price of the mixture.

Let

Let x = the price of a gallon of the mixture,

$\therefore x \times \overline{a + b}$ = price of the mixture,

$\therefore x \times \overline{a + b} = ma + nb$

$$\therefore x = \frac{ma + nb}{a + b}.$$

Suppoſe 50 gallons, at five ſhillings the gallon, to be mixed with 30 gallons, at ten ſhillings;

$$\text{then } x = \frac{50 \times 5 + 30 \times 10}{50 + 30} = \frac{550}{80} = 6s.\ 10\tfrac{1}{2}d.$$

From the equation $x \times \overline{a + b} = ma + nb$; if any four out of the five numbers x, a, b, m, n, are given, the other may be found.

QUESTIONS FOR THE LEARNER TO SOLVE BY HIMSELF.

WHAT number is that, which, being multiplied by three, makes, with ſeven added to the product, the ſum 100? Anſ. 31.

What number is that, which, being divided by 4, gives a quotient, from which, if you take away 9, the remainder ſhall be 20? Anſ. 116.

A field produced eight quarters of wheat, which made only a third part more than what had been ſown: How much wheat was ſown? Anſ. 6 quarters.

Two hundred and ten pounds were given to two perſons, to A half as much as B: What ſum was given to each? Anſ. A had 70 pounds, B 140 pounds.

Out of a caſk of liquor, 63 gallons were ſold to two perſons, to the firſt one-third, and to the ſecond one-ſixth of what the caſk contained: What did the caſk hold? Anſ. 126 gallons.

A perſon diſtributing ſome money among beggars, wanted eight pence to give them three pence a piece; he therefore gave to each two pence, and had three pence remaining: How many beggars were there? Anſ. 11.

ON SIMPLE EQUATIONS HAVING TWO OR MORE UNKNOWN NUMBERS.

Sometimes there appears in a ſimple equation more than one unknown number, and their values may be diſcovered by means of as many ſimple equations as there are unknown numbers. Thus, ſuppoſe the ſum and difference of two unknown numbers to be given, let it be required to find the numbers. Let a = the ſum, and b the difference of the two unknown numbers.

Then $x + y = a \quad \therefore \quad x = a - y$

and $x - y = b \quad \therefore \quad x = b + y$

$\therefore \quad a - y = b + y$

$\therefore \quad a - b = 2y$

$\therefore \quad y = \frac{a - b}{2}.$

and $x = a - \frac{a - b}{2}.$

Suppoſe,

Suppoſe, of three unknown numbers, the ſum of the firſt and ſecond, of the firſt and third, and of the ſecond and third to be given, what are the numbers?

Let the numbers be x, y, z.

then $x + y = a \quad \therefore x = a - y$

$x + z = b \quad \therefore x = b - z$

$y + z = c \quad \therefore y = c - z$.

Hence $a - y = b - z$

$\therefore a - b + z = y = c - z$

$\therefore 2z = c - a + b$

$$\therefore z = \frac{b + c - a}{2}$$

$$\therefore y = c - \frac{b + c - a}{2}$$

$$\text{and } x = b - \frac{b + c - a}{2}.$$

In theſe, and ſimilar caſes, two values of one unknown number are obtained, by which means two terms are made equal, which contain two other unknown numbers; and hence a value of one of theſe unknown numbers is obtained, which may be compared with its value derived from the third equation, and hence is derived an equation containing only one unknown number. If there had been more unknown numbers, there would have been more ſimple equations, but the proceſs would have been equally eaſy.

INSTANCES.

To find two numbers whoſe ſum is s, and their proportions as a to b.

Let

Let x and y be the two numbers;

then $x + y = s \quad \therefore x = s - y$

and $x : y :: a : b \quad \therefore x = \frac{ay}{b}$

$\therefore s - y = \frac{ay}{b}$

$\therefore bs = ay + by = y \times \overline{a + b}$

$\therefore y = \frac{bs}{a + b}$

$\therefore x = s - y = s - \frac{bs}{a + b}$.

Sometimes the two unknown numbers may be diſcovered in a ſhorter manner. Thus, in a company of 18 perſons, men and women, the reckoning was 9*l.* 18*s.* for which each perſon paid as many ſhillings as there were men in company: How many were there?

Let x = the number of men,
y = the number of women.

$\therefore xx$ ſhillings = ſum paid by the men,
xy ſhillings = women.

1ſt. $x + y = 18$
2d. $xx + xy = 198$ ſhillings (= 9*l.* 18*s.*)

Divide 2d by 1ſt $\left\{ \frac{xx + xy}{x + y} \right.$ or $x \times \frac{x + y}{x + y} = \frac{198}{18}$

$\therefore x = 11$
and $y = 7$.

Eight

Eight yards of cloth coſt 62 ſhillings, of which ſome coſt 7 ſhillings, and the others 9 ſhillings a yard: How many yards were there at each of theſe prices?

Let x yards coſt 9s. and y yards 7s.

$$\therefore x + y = 8$$
$$9x + 7y = 62s.$$

Multiply 1ſt by 7.

$$7yx + 7y = 56.$$

from $9x + 7y = 62$

take $7x + 7y = 56.$

$$\therefore 2x = 6$$
$$\therefore x = 3$$
$$\therefore y = 8 - x = 8 - 3 = 5.$$

There are two numbers, of which the greater is to the leſs, as their ſum to 45, and as their difference to 9: What are the numbers?

Let x = one number,
y = the leſs number.

$$\therefore x : y :: x + y : 45$$
and $x : y :: x - y : 9.$

by the 1ſt prop. $45x = xy + yy$

2d. $9x = xy - yy$

Sum of the two equ^s. $54x = 2xy.$

$$\therefore y = \frac{54}{2} = 27$$
$$\therefore 9x = 27x - 27.27$$
$$\therefore x = 3x - 3.27$$
$$\therefore 3.27 = 3x - x$$
$$\therefore 81 = 2x$$
$$\text{and } x = \frac{81}{2} = 40\tfrac{1}{2}.$$

ON

ON FRACTIONS.

ANY number may be considered as a whole, or as a part of another number. Thus 5, considered in itself, is a whole number; considered with relation to 20, it is a part, namely, a fourth part. Five shillings make a crown, and sums may be computed by crowns as well as pounds; but also five shillings, or a crown, may be considered as a fourth part of a pound.

A fraction, or, according to the derivation of the word, a broken number, is a number denoting a part or parts of some whole, and it is written down, by placing the parts into which a whole number is divided below a straight line; and the number of those parts to be taken, above the line. Thus $\frac{1}{4}$ is a fraction, the number 4 denoting that a whole number is divided into 4 parts, and 1 denoting that one of these parts only is to be considered. $\frac{3}{4}$ means 3 of those 4 parts or three-fourths; $\frac{7}{4}$ means 7 of those 4 parts or seven-fourths; $\frac{m}{n}$ means, that a whole number is divided into n parts, and that m of these parts are to be considered in the proposed question. Hence $\frac{m}{n}$ is the same as $m \times \frac{1}{n}$; $\frac{3}{4}$ the same as $3 \times \frac{1}{4}$; $\frac{7}{2}$ the same as $7 \times \frac{1}{2}$.

A fraction consists of two parts, the upper and the lower part. The upper part may be called the numberer, or the numerator, as it gives the number of the parts to be considered: the lower part may be called the denoter, or denominator, as it denotes the parts into which the

whole number is to be divided. But perhaps it will be better to make uſe always of the ſimpleſt terms, the upper and the lower parts of a fraction.

When the upper part is leſs than the lower part, the fraction is called a proper fraction. $\frac{1}{5}$ is a proper fraction, for it means a fifth part of ſome whole, that is, it is a broken number.

When the upper part is greater than, or equal to, the lower part, the fraction is called an improper fraction; for though it has the form of a fraction, it is, ſtrictly ſpeaking, either a mixed number or a whole. Thus $\frac{7}{3}$ is an improper fraction; for $\frac{7}{3}$ means ſeven-thirds of a whole. But if a whole number is divided into three parts, three of thoſe parts make up the whole number, and ſix of thoſe parts make double that number. Therefore ſeven of theſe parts make double the whole number, and one of theſe parts; that is, $\frac{7}{3} = 2 + \frac{1}{3}$. Again $\frac{3}{3}$ is an improper fraction, for it is equal to unity.

A fraction is always equal to the number found by dividing the upper part by the lower; or in caſe ſuch a diviſion cannot take place, the line between the two parts denotes that the upper is to be divided by the lower part. Let $\frac{a}{b}$ repreſent any fraction equal to m, then $\frac{a}{b}$ means that a is to be multiplied into $\frac{1}{b}$ and $a \times \frac{1}{b} = m$.

Multiply both ſides by b.

$$\therefore a \times \frac{1}{b} \times b = bm$$

but

but b times $\frac{1}{b} = 1$

$\therefore a = bm$

Divide both ſides by b.

$\therefore m = \frac{a}{b}$.

$\frac{3}{4} = 3 \times \frac{1}{4}$, or $3 \div 4$.

$\frac{1}{4}$ of a pound is 5 ſhillings; therefore three-fourths of a pounds is 15 ſhillings, or three pounds divided by four, that is, $\frac{60 \text{ ſhillings}}{4} = 15$ ſhillings.

If both the upper and lower parts of a fraction are multiplied by the ſame number, its value is not altered.

Let $\frac{a}{b} = m$.

Multiply both ſides of the equation by a, and divide by a; then $\frac{aa}{ba} = m \times \frac{a}{a}$, but $\frac{a}{a} = 1$ $\therefore m \times \frac{a}{a} = m$ $\therefore \frac{aa}{ba} = m$, or the value of the fraction remains the ſame.

Multiply both parts of $\frac{2}{3}$ by four, then $\frac{2}{3} = \frac{8}{12}$. Two thirds of a pound = twice 6s. 8d. and $\frac{8}{12}$ = 8 times 1s. 8d. $\therefore \frac{2}{3}l. = 13s.\ 4d. = \frac{8}{12}l.$

ADDITION OF FRACTIONS.

LET $\frac{a}{b} = m$ and $\frac{c}{d} = n$, therefore the sum of $\frac{a}{b}$ and $\frac{c}{d}$ is equal to the sum of m and n.

$$\frac{a}{b} + \frac{c}{d} = m + n.$$

Multiply both sides by b and d.

$$\frac{abd}{b} + \frac{cbd}{d} = \overline{m + n} \cdot b \cdot d.$$

$$\therefore ad + cb = \overline{m + n} \cdot b \cdot d$$

$$\therefore m + n = \frac{ad + cb}{b \times d}.$$

Rule. Multiply the upper part of each fraction into the lower part of the others, and divide the sum of the products by the product of all the lower parts.

Add together $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

Let $\frac{a}{b} = m$, $\frac{c}{d} = n$, $\frac{e}{f} = p$.

$$\therefore \frac{a}{b} + \frac{c}{d} + \frac{e}{f} = m + n + p.$$

$$\therefore adf + cbf + ebd = \overline{m + n + p} \times b \cdot d \cdot f.$$

$$\therefore m + n + p = \frac{adf + cbf + ebd}{b \times d \times f.}$$

Add

Add together $\frac{3}{4}$ and $\frac{4}{5}$.

$$\frac{3}{4} + \frac{4}{5} = \frac{15 + 16}{20} = \frac{31}{20} = 1 + \frac{11}{20}.$$

three-fourths of a pound = 15 ſhillings,
four-fifths of a pound = 16 ſhillings,
$\frac{3}{4} + \frac{4}{5}$ = 31 ſhillings, or $\frac{31}{20}$ of a pound.

Add together $\frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$. Anſ. $3\frac{1}{20}$.
Add together $\frac{1}{2} + \frac{1}{3} + \frac{9}{10}$. Anſ. $1\frac{11}{15}$.
Add together $\frac{3a}{b} + \frac{10c}{6d} + \frac{9a}{5b}$. Anſ. $\frac{90ad + 50bc + 54ad}{30bd}$.

If a whole number is to be added to a fraction, it is to be conſidered as the upper part of a fraction, whoſe lower part is unity, and the addition is to be made as before.

Add a and $\frac{b}{c}$ together.

$$\frac{a}{1} + \frac{b}{c} = m$$
$$\therefore ac + b = cm$$
$$\therefore \frac{ac + b}{c} = m.$$

Add together $7\frac{2}{3}$ and $\frac{4}{5}$.

$$\frac{7\times3\times5+2\times5+3\times4}{3\times5} = \frac{105+10+12}{15} = \frac{127}{15} = \text{ſum.}$$

Add together $7\frac{2}{3}$, $4\frac{3}{5}$, $6\frac{7}{8}$. Sum $19\frac{17}{120}$.

In the above, and ſimilar inſtances, the whole numbers may be firſt added together, 7 + 4 + 6 = 17, and the

fractions may be added to the number thus found, by which the ſum may be found with leſs trouble.

Add together $a, \frac{m}{n}, \frac{c}{d}$, and b. Sum $a + b + \frac{md + cn}{nd}$.

Add together $\frac{4a}{3}, \frac{5b}{6}, 7\frac{2}{3}$. Sum $7 + \frac{72a + 45b + 36}{54}$.

SUBTRACTION.

Let $\frac{a}{b} = m, \frac{c}{d} = n$

then $\frac{a}{b} - \frac{c}{d} = m - n$.

Multiply both ſides by $b \cdot d$.

$$\therefore \frac{abd}{b} - \frac{cbd}{d} = \overline{m - n} \times b \cdot d$$

$$\therefore ad - cb = \overline{m - n} \times b \cdot d$$

$$\therefore m - n = \frac{ad - cb}{b \times d}.$$

But $m - n$ = the difference of $\frac{a}{b}$ and $\frac{c}{d}$.

Rule. Multiply the upper part of each fraction into the lower part of the other, and the difference of theſe products, divided by the product of the lower parts, is the difference of the two fractions.

Take away $\frac{3}{4}$ from $\frac{4}{5}$.

$$\frac{16-15}{20} = \frac{1}{20} = \frac{4}{5} - \frac{3}{4}.$$

$\frac{3}{4}$ of a pound = 15 ſhillings,
$\frac{4}{5}$ of a pound = 16 ſhillings,
Difference = 1 ſhilling = $\frac{1}{20}$ of a pound.

Take away $\frac{4a}{3b}$ from $\frac{6a}{7c}$. Diff. $\frac{18ab - 28ac}{21bc}$.

Take away $\frac{3ab}{7d}$ from $\frac{9c}{12e}$. Diff. $\frac{63cd - 36abe}{84ed}$.

If a fraction is to be taken away from a whole number, or a whole number from a fraction, the whole number is to be conſidered as the upper part of a fraction, whoſe lower part is unity.

Take away $\frac{b}{c}$ from a.

Let $a - \frac{b}{c} = m$
$\therefore ac - b = cm$
$\therefore m = \frac{ac - b}{c}$.

As a has no lower part, b in the upper part of the fraction remains without a co-part.

Take away $\frac{2}{5}$ from 7.

$$7 - \frac{2}{5} = \frac{35 - 2}{5} = \frac{33}{5} = 6\frac{3}{5}.$$

Take away $4\frac{1}{6}$ from $7\frac{2}{3}$. Anſ. $2\frac{3}{6}$.

Add together $7\frac{2}{3}$ for an improper fraction, from which take away $4\frac{5}{8}$ added together.

Take away $8\frac{3}{4}$ from $9\frac{1}{2}$. Anſ. $\frac{3}{4}$.

Take away $16a + \frac{mo}{b}$ from $17a + \frac{c}{d}$. Anſ. $a + \frac{cb - mod}{bd}$.

MULTIPLICATION.

Let $\frac{a}{b} = m$, and $\frac{c}{d} = n$

$\therefore a = m.b$, and $c = n.d$

$\therefore a \times c = m.b.n.d.$

Divide both ſides by $b.d.$

$$\therefore \frac{a \times c}{b \times d} = m \times n$$

but $m \times n = \frac{a}{b} \times \frac{c}{d}$

$$\therefore \frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

Rule. Multiply the upper parts together for a new upper part, and the lower parts together for a new lower part. Then the new fraction is the product of the multiplication of the propoſed fractions: or the product of any number of fractions multiplied together is equal to the product of their upper parts divided by the product of their lower parts.

Multiply $\frac{3}{4} \times \frac{4}{5}$. Prod. $\frac{12}{20}$.

Multiply $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f}$. Prod. $\frac{ace}{bdf}$.

If a whole number is to be multiplied into a fraction, it is to be considered as an upper part, having a lower part unity.

Multiply $a \times \frac{b}{c}$

let $\frac{b}{c} = m$

$\therefore \frac{ab}{c} = am.$

Multiply $5 \times \frac{4}{3} \times \frac{2}{7}$. Prod. $\frac{40}{21} = 1\frac{19}{21}$.

Multiply $8a \times \frac{3b}{c} \times 9$. Prod. $\frac{216ab}{c}$.

DIVISION.

Let $\frac{a}{b} = m$, $\frac{c}{d} = n$.

$\therefore a = bm$

and $c = dn$

$\therefore \frac{a}{c} = \frac{bm}{dn}$

$\therefore \frac{ad}{bc} = \frac{m}{n}$.

But

$$\text{But } \frac{m}{n} = \frac{a}{b} \div \frac{c}{d}$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

Rule. Multiply the upper part of the greater fraction into the lower part of the leaſt, and the upper part of the leaſt into the lower part of the greateſt fraction, and the product from the firſt multiplication, divided by the product from the other, gives the reſult of dividing the firſt or greateſt fraction by the other.

Divide $\frac{4}{5}$ by $\frac{3}{4}$. Reſ. $1\frac{1}{15}$.

Divide $\frac{7}{8}$ by $\frac{6}{7}$. Reſ. $\frac{49}{48}$.

Divide $\frac{15}{17}$ by $\frac{3}{5}$. Reſ. $1\frac{3}{17}$.

If a whole number is to be divided by a fraction, it is to be conſidered as an upper part, having a lower part unity.

Divide a by $\frac{b}{c}$.

$$\text{Let } \frac{b}{c} = m$$

$$\therefore b = cm$$

$$\therefore \frac{a}{b} = \frac{a}{cm}$$

$$\therefore \frac{ac}{b} = \frac{a}{m}$$

$$\therefore a \div \frac{b}{c} = \frac{ac}{b}.$$

Divide

Divide $8a$ by $\frac{4cd}{e}$. Ref. $\frac{2ae}{cd}$.

Divide $20abc$ by $\frac{5acd}{4ef}$. Ref. $\frac{16bef}{}$.

If the upper and lower parts of a fraction are divided by the ſame number, the fraction remains the ſame.

Let $\frac{a}{b} = m$

then $a = bm$.

Divide both ſides by n.

$$\therefore \frac{a}{n} = \frac{b}{n} \cdot m$$

and $$\frac{a}{n} \div \frac{b}{n} = m$$

but $$\frac{a}{n} \div \frac{b}{n} = \frac{an}{bn} = \frac{a}{b}.$$

It is uſeful ſometimes to bring a fraction into lower terms, and at other times to raiſe the terms of a fraction without altering its value; and this is done by dividing or multiplying the upper and lower parts of the fraction by the ſame number.

Let $\frac{a}{b}$ repreſent any fraction.

Suppoſe $a = cde$

$b = defg$

then $\frac{a}{b} = \frac{cde}{defg} = \frac{c}{fg}$.

Hence,

Hence, if in any fraction a number is found which will divide both parts, the upper and the lower parts may be both diminiſhed, whilſt the value of the fraction remains unaltered.

It is required to bring the fraction $\frac{24}{72}$ into lower terms.

$$24 = 2 \cdot 3 \cdot 4$$
$$72 = 2 \cdot 3 \cdot 4 \cdot 3$$
$$\therefore \frac{24}{72} = \frac{2 \cdot 3 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 3} = \frac{1}{3}.$$

Or by ſtriking out only one or two of the figures, in each part, the fraction might have been made

$\frac{12}{36}$, or $\frac{6}{18}$, or $\frac{2}{6}$, or $\frac{1}{3}$.

A fraction is brought into its loweſt terms when the upper and lower parts are divided by the greateſt common part of each.

$$\frac{a}{b} = \frac{cde}{defg}.$$

de is here the greateſt common part of each; then $\frac{c}{fg}$ is the fraction $\frac{a}{b}$ in its loweſt terms.

$$\frac{35}{63} = \frac{7 \times 5}{7 \times 9}.$$

7 is the greateſt part common to 35 and 63,
$\therefore \frac{5}{9}$ is the fraction in its loweſt terms.

To bring the fraction into the loweſt terms, divide both parts by the leaſt, and if there is no remainder, the fraction is in its loweſt term.

In

In $\frac{a}{b}$ let $b = am$.

Divide both parts by a,

then the fraction becomes $\frac{1}{m}$.

If there is a remainder, divide that remainder by the firſt divider, and ſo on, till there is no remainder, and the laſt divider will be the greateſt common part of the two terms in the fraction; and by dividing them by this common part, the fraction is reduced to its loweſt terms.

In the fraction $\frac{a}{b}$ let b be the greater term.

$$\begin{array}{r} a \,)\, b \;(\, m \\ ma \\ \hline c \,)\, a \;(\, n \\ nc \\ \hline d \,)\, c \;(\, p \\ pd \\ \hline 0. \end{array}$$

In dividing b by a, let c be the difference between b and ma, d the difference between nc and a, and let $pd = c$.

then $a = nc + d = npd + d = d \times \overline{np+1}$

$b = ma + c = mnpd + md + pd = d \times mnp + m + 1$.

therefore d is a common part of both a and b, and if a and b are both divided by d, the fraction becomes $\frac{np+1}{mnp+m+1}$, which

which is in its loweſt terms, as there is no common part to the two terms.

For $a \div d = p + 1$

$b \div d = mnp + m + 1.$

Therefore the fraction, after dividing both parts by d, becomes $\frac{np+1}{mnp+m+1}$.

Bring the fraction $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$ to its loweſt terms.

$$a^2 - b^2 = \overline{a+b} \times \overline{a-b}$$

$$a^2 - 2ab + b^2 = \overline{a-b} \times \overline{a-b}$$

$$\therefore \frac{a^2 - b^2}{a^2 - 2ab + b^2} = \frac{\overline{a+b} \times \overline{a-b}}{\overline{a-b} \times \overline{a-b}} = \frac{a+b}{a-b}$$

Bring the fraction $\frac{16}{98}$ into its loweſt terms.

16) 98 (6
96

2) 16 (8
16

0

Divide 16 and 98 by 2, and the fraction becomes $\frac{8}{49}$.

When in any fraction a number raiſed to different powers is ſeen in both parts, the fraction is reduced to lower terms, by ſtriking out the leaſt power, and ſubtracting its index from the index of the other power.

$$\text{Thus } \frac{a^3 b}{a^5} = \frac{b}{a^{5-3}} = \frac{b}{a^2}$$

for $a^3 = aaa$, and $a^5 = aaaaa$; the common parts of each therefore are aaa, which are to be ſtruck out in both terms, and $a^2 = aaaaa$, after the three a's are ſtruck out. In general, powers are multiplied together by adding their indexes, and divided by ſubtracting the leſs from the greater.

$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}$$

$$\frac{a^3}{a^2} = \frac{aaa}{aa} = a = a^{3-2} = a.$$

Note. This is merely done for the ſake of conciſeneſs: but there is one exception; if two powers have the ſame index, this taking away may lead into errour. $\frac{a^2}{a^2} = a^{2-2} = a^0$, which ſhews that there is no index to a, and conſequently no power whatever of a produced, by dividing a^2 by a^2. We know that a^2 divided by a^2, or any quantity divided by itſelf is equal to 1: but we muſt not therefore ſay, that a^0 is equal to unity, as no number whatever can be denoted by a^0.

ON

ON DECIMAL FRACTIONS.

A DECIMAL fraction is one whose lower part is ten, or some power of ten. Thus $\frac{1}{10}$, $\frac{2}{10^2}$, $\frac{34}{10^4}$ are decimal fractions.

Instead of writing down the lower part, these fractions are generally written down with a comma to the left of the upper part, which denotes that the numbers following this comma belong to the upper part of a fraction, whose lower part is a power of ten, with an index equal to the number of figures following the comma. Thus ,1 is a fraction whose lower part is ten, there being only one figure after the comma: ,02 is a fraction whose upper part is 2, and lower part 10^2 or 100, there being two figures after the comma: ,0034 is a fraction, whose upper part is 34, and lower part 10^4, or 10000, there being four figures after the comma.

ADDITION OF DECIMAL FRACTIONS.

LET ,m and ,n represent any two decimal fractions or numbers made up of a whole number and a decimal fraction; ,m being equal to $\frac{m}{10^p}$, and ,n being equal to $\frac{n}{10^q}$ p also being greater than q. Then $,m + ,n = \frac{m}{10^p} + \frac{n}{10^q} = \frac{n.10^p + m.10^q}{10^{p+q}} = 10^q \times \frac{n.10^{p-q} + m}{10^{p+q}} = \frac{n.10^{p-q} + m}{10^p}$.

Multiply

Rule. Multiply the upper part of the decimal, which had the ſmalleſt under part, into the power of ten, whoſe index is equal to the difference between the indexes of the two lower parts *. Add to it the other upper part, and divide the ſum by the greateſt power of ten in the two fractions to be added together.

Add together ,123 and ,34.

$$,123 = \frac{123}{10^3} \quad ,34 = \frac{34}{10^2}$$

$$34 \times 10^{3-2} = 34 \times 10 = 340$$

$$\frac{123 + 340}{10^3} = \frac{463}{10^3} = ,463.$$

Add together ,009 and ,078.

$$,009 = \frac{9}{10^3} \quad ,78 = \frac{78}{10^3}$$

$$,009 + ,078 = \frac{9}{10^3} + \frac{78}{10^3} = \frac{87}{10^3} = ,087.$$

In writing down a decimal fraction, cyphers are placed between the comma and the upper part of the fraction, to denote the number of figures on which the power of 10, in the under part, depends.

$$\frac{34}{10^5} = ,00034.$$

The lower part of $\frac{34}{10^5}$ is 10^5. There muſt be therefore

* If this difference is equal to nothing, the under parts of the two fractions are equal, this multiplication does not take place, and the two upper parts are to be added together.

five figures after the comma, and the three cyphers do not increaſe or diminiſh at all the upper part, but ſimply ſhew, that in the lower part 10 is raiſed to the fifth power.

Write down $\frac{2}{10^2}$; $\frac{84}{10^5}$; $\frac{33}{10^6}$; $\frac{345}{10^8}$.

Inſtead of making the multiplication preſcribed in the above rule, decimals may be added together, by placing them, one under the other, and proceeding as in common addition; care being taken only that the figures, at equal diſtances from the comma, ſhould be in the ſame line.

Add together ,123 and ,34.

,123
,34
Sum ,463

By the rule 34 was to have been multiplied into ten, and if we had done it in this inſtance, we ſhould not have made any alteration in the ſum

,123
,340
Sum ,463

The reaſon of this is evident, both from the rule itſelf, and the conſideration that 340 is the ſame as $\frac{340}{10^3}$ or $\frac{34}{10^2}$. The placing of a cypher after the decimal is the ſame as multiplying the upper and lower part by 10, by which multiplication the fraction, as has been before obſerved, is not increaſed.

Add

Add together ,789 ,4998 ,00365. Sum 1,29245.

Add together ,00379 ,16 ,3 ,12. Sum ,58379.

Add together ,05 ,789, ,261. Sum 1,1.

If whole numbers are to be added together with theſe fractions, they are ſet down as in common addition, the comma diſtinguiſhing between the whole numbers and the fractions; the numbers on the left of the comma being whole numbers, thoſe on the right being fractions.

Add together 7,03 : 12,9 : 265,73104.

	7,03
	12,9
	265,73104
Sum	285,66104

In the above inſtances the ſum of the fractions is greater than unity, conſequently the unit is added to the units in the whole numbers.

SUBTRACTION OF DECIMAL FRACTIONS.

Let $,m$ and $,n$ repreſent, as before, any two decimal fractions, whoſe lower parts are 10^P and 10^q.

Then $,n - ,m = \frac{n}{10^q} - \frac{m}{10^P} = \frac{n.10^P - m.10^q}{10^{P+q}} = 10^q \times \frac{n.10^{P-q} - m}{10^{P+q}} = \frac{n\,10^{P-q} - m}{10^P}$.

In the above caſe 10^q is leſs than 10^p, but it might have been greater, and ſtill the fraction $\frac{n}{10^q}$ might have been greater than $\frac{m}{10^p}$. Let q then be greater than p.

$$,n - ,m = \frac{n}{10^q} - \frac{m}{10^p} = \frac{n.10^p - m.10^q}{10^{p+q}} = 10^p \times \frac{n - m.10^{q-p}}{10^{p+q}} = \frac{n - m.10^{q-p}}{10^q}.$$

Rule. Multiply the upper part of the fraction, which has the feweſt decimal places, into that power of ten whoſe index is equal to the difference of the decimal places in both fractions, and then take away the leſs number from the greater; the number of decimal places in the remainder will be the ſame as in that fraction which had the greateſt number of decimal places.

Take away ,023 from ,19865.

$$,19865 = \frac{19865}{10^5} \quad ,023 = \frac{23}{10^3}$$

$$\therefore ,19865 - ,023 = \frac{19865}{10^5} - \frac{23}{10^3} = \frac{19865 - 23.10^2}{10^5} = \frac{19865 - 2300}{10^5} = \frac{17565}{10^5} = ,17565.$$

Inſtead of making this multiplication, the decimal fractions may be written down with the comma before them, and then the leſs may be taken from the greater, as in common ſubtraction; care being taken, as before, that the numbers at equal diſtances from the comma may be placed in the ſame line, and the number of decimal places be

be made equal in both places, by placing cyphers after that which has the feweſt decimal places.

Take away ,023 from ,19865.

From	,19865
take	,02300
Remainder	,17565.

,023 is the ſame as ,023 $\times \frac{10^2}{10^2}$ or ,02300; and ,02300 and ,19865 have the ſame power of 10 in the under parts. In the whole numbers, in taking away 2300 from 19865, they are to be ſo placed, that numbers of the ſame power may be placed under each other:

19865
2300.

The ſame thing is done in the decimals, by placing them under each other, according to their diſtance from the comma.

Take away ,569 from ,84.

,840
,569
,271

Take away 12,584 from 24,21.

24,210
12,584
11,626.

Apply what was said on addition of mixed numbers (page 67) to the instance above.

From 54,57 take 8,999. Rem. 45,571.
From 6,7986 take 5,89999. Rem. ,89861.
From 27,493 take 26,9879. Rem. ,5051.

MULTIPLICATION OF DECIMAL FRACTIONS.

Let $,m$ and $,n$ represent, as before, any two decimal fractions $,m = \frac{m}{10^p}$ and $,n = \frac{n}{10^q}$.

$$\therefore ,m \times ,n = \frac{m}{10^p} \times \frac{n}{10^q} = \frac{mn}{10^{p+q}}.$$

Rule. Multiply the two upper parts together, and the product will have a number of decimal places equal to the sum of the decimal places in both fractions.

Multiply ,54 into ,325.

$$,54 \times ,325 = \frac{54}{10^2} \times \frac{325}{10^3} = \frac{17650}{10^5} = ,17550.$$

,325
,54
1300
1625
,17550.

Multiply

Multiply 3,25 into 5,4, and the product becomes 17,550.

Multiply 32,4 into 5,4. The product is 17,55.

Multiply ,00325 into ,54. The product is ,0017550.

DIVISION OF DECIMAL FRACTIONS.

Let $,m$ and $,n$ represent, as before, any two decimals, $,m = \frac{m}{10^p}$ and $,n = \frac{n}{10^q}$.

Then $,m \div ,n = \frac{m}{10^p} \div \frac{n}{10^q} = \frac{m.10^q}{n.10^p} = \frac{m}{n} \times \frac{10^q}{10^p} = \frac{m}{n} \times 10^{q-p}$ or $\frac{m}{n} \times \frac{1}{10^{p-q}}$, according as q is greater or less than p.

Rule. Divide the greater upper part by the less, and the result will have a number of decimal places equal to the difference of the number of decimal places in the two fractions. Or if the number of decimal places in the divider is greater than that in the fraction to be divided, the result is to be multiplied into that power of 10, whose index is equal to the difference of the numbers of decimal places in the two fractions.

Divide ,0546 by ,00003.

$$,0546 \div ,00003 = \frac{546}{10^4} \div \frac{3}{10^5} = \frac{546}{3} \times \frac{10^5}{10^4} = \frac{546}{3} \times 10 = 1820.$$

Divide ,5 by ,003.

$$5 \times 10^{3-1} = 500.$$

$$\frac{500}{3} = 166\tfrac{2}{3}.$$

Divide 7,5344 by ,4.

$$\frac{7{,}5344}{{,}4} = \frac{75344}{10^4} \div \frac{4}{10} = \frac{75344}{4} \times \frac{1}{10^3} = 18{,}836.$$

Divide ,54632 by ,53. Reſult $1{,}030\frac{42}{53}$.

TO REDUCE A COMMON TO A DECIMAL FRACTION.

Let $\frac{a}{b}$ repreſent any fraction, b not being equal to 10 or any power of 10; and let $,x$ repreſent the decimal fraction.

$$\therefore \frac{a}{b} = ,x = \frac{x}{10^p}$$

$$\therefore x = \frac{a.10^p}{b}.$$

Rule. Multiply the upper part of the given fraction by ten, raiſed to a power ſo great, that the product may be diviſible by the lower part without a remainder; and the reſult of this diviſion gives the decimal, the number of whoſe decimal places is equal to the index of ten, into which the upper part was multiplied.

Reduce $\frac{3}{4}$ into a decimal fraction.

$$\frac{3}{4} = ,x = \frac{x}{10^2}$$

$$\therefore x = \frac{3 \times 10^2}{4} = \frac{300}{4} = \frac{75}{1^2} \therefore ,x = ,75.$$

If

If 10^3 or 10^4 had been taken, the reſult would have been ſtill equal to $\frac{3}{4}$; for it would have ,750 or ,7500, both which are equal to ,75.

The product of the upper part of the given fraction, multiplied into any power of ten whatever, cannot always be divided by the lower part of the given fraction, without a remainder; in which caſe the decimal cannot be accurately given, but the farther the diviſion is carried, the nearer does the decimal approach to equality with the given fraction.

Reduce $\frac{1}{3}$ to a decimal fraction.

$$3 \overline{)\ 1,000000000}$$
$$,333333333$$

In this caſe the diviſion may be carried on without end; but a ſmall number of places will be ſufficient for moſt calculations. Suppoſe one is taken, then $,3 = \frac{3}{10}$; and $\frac{3}{10}$ is leſs than $\frac{1}{3}$ by $\frac{1}{3.10}$. If two places are taken, $33 = \frac{33}{100}$, which is leſs than $\frac{1}{3}$ by $\frac{1}{3.100}$. If three places are taken, $,333 = \frac{333}{1000}$, which is leſs than $\frac{1}{3}$ by $\frac{1}{3.1000}$. And in general, if n places are taken, the decimal ,333 .... to n places $= \frac{33 \text{ .... to } n \text{ places}}{3.10^n}$, which is leſs than $\frac{1}{3}$ by $\frac{1}{3.10^n}$.

Reduce $\frac{5}{9}$ to a decimal with ſix places, and find the difference between it and the given fraction.

Anſ. Dif. $\frac{5}{9.10^6}$.

Reduce

Reduce $\frac{1}{9}$ to a decimal with four places, and find the difference between it and the given fraction.

Anſ. Dif. $\frac{1}{9.10^4}$.

Inſtead of the ſame number being repeated in the decimal, a ſeries of numbers will recur over and over again; and hence theſe decimals, as well as the former, are called recurring decimals; and when a ſeries of numbers recurs, the decimal is called a circulating decimal.

Reduce $\frac{1}{11}$ to a decimal fraction.

11) 1,00000000 - - -

,09090909 - - -

In the above inſtances 09 recurs continually; and in this caſe there is a material difference between the taking of an even and an odd number of places.

If two places are taken, the difference between $\frac{1}{11}$ and ,09 or $\frac{9}{100}$ is $\frac{1}{1100}$; if three places are taken, the difference between $\frac{1}{11}$ and ,090 or $\frac{90}{1000}$ is $\frac{1}{1100}$, the ſame as when two places only were taken. If four places are taken, the difference between $\frac{1}{11}$ and ,0909 or $\frac{909}{10000}$ is $\frac{1}{11.100.000}$. Let n be an even number of places, and the difference between $\frac{1}{11}$ and ,0909 .... to n places is $\frac{1}{11.10^n}$. If n is an odd number, the difference between $\frac{1}{11}$ and ,090909 to n places is $\frac{1}{11\ 10^{n-1}}$. It is evident, therefore, that, in applying theſe decimals to practice, an even number ſhould be taken.

Reduce

Reduce $\frac{3}{7}$ to a decimal.

$$7\,)\,3{,}0000000000$$

$$,428571428571$$

Here the decimal returns after every ſix places; and, though by taking a greater number of places, the decimal approaches nearer and nearer to the given fraction, yet theſe approaches are made very irregularly, as the learner will ſee, by finding the difference between $\frac{3}{7}$ and the decimal having ſucceſſively one, two, three, four, &c. places.

A decimal being given, the common fraction equal to it may be found, having any given upper or lower part.

Let $,m$ equal the decimal, and $\frac{a}{b}$ the common fraction.

$$\text{then } \frac{m}{10^P} = \frac{a}{b}$$

$$\therefore a = \frac{bm}{10^P}$$

$$\text{or } b = \frac{a.10^P}{m}.$$

Rule. For the upper part multiply the given lower part into the decimal. For the lower part divide the given upper part by the decimal.

Find a common fraction equal to ,75 whoſe lower part is four.

$$,75 = \frac{a}{4} = \frac{75}{100}$$

$$\therefore a = \frac{75.4}{100} = \frac{300}{100} = 3.$$

$\therefore$ the common fraction is $\frac{3}{4}$.

Find

Find a common fraction equal to ,75 whose upper part is 5.

$$,75 = \frac{5}{b} = \frac{75}{100}$$

$$\therefore b = \frac{500}{75} = 6,666\text{-----}$$

$$\therefore ,75 = \frac{5}{6,666\text{-----}}$$

In this case, the lower part is a mixed number.

To find a common fraction or whole number, to which a recurring decimal with several series of recurring numbers is nearly equal.

Let ,428571 428571 428571 ----- 428571 be the recurring decimal, and let it be equal to S.

$$\therefore S = ,428571\ 428571\ 428571\ \text{-----}\ 428571.$$

Multiply both sides by 1000000.

$$\therefore 1000000S = 428571,428571428571\ \text{----}\ 428571000000.$$

But the decimal in this equation is very nearly equal to the given decimal, that is to S.

$$\therefore 1000000\ S = 428571 + S \text{ nearly.}$$

$$\therefore 1000000\ S - S = 428571 \text{ nearly.}$$

$$\therefore S = \frac{428571}{999999} \text{ nearly.}$$

Bring this fraction to its loweſt terms.

$$\begin{array}{r|l} 428571 & 999999 \;(\; 2 \\ & 857142 \\ \hline 142857 & 428571 \;(\; 3 \\ & 428571 \\ \hline & 00000 \end{array}$$

$$\begin{array}{r|l} 142857 & 999999 \;(\; 7 \\ & 999999 \\ \hline & 000000 \end{array}$$

$\therefore S = \frac{2}{7}$ nearly.

Find a common fraction to which ,123 123 123 is nearly equal.

$S = ,123\ 123\ 123.$

Multiply both ſides by 1000.

$\therefore 1000\ S = 123,\ 123\ 123,$

but ,123 123 is nearly equal to the given decimal.

$\therefore 1000\ S = 123 + S$ nearly.

$\therefore 1000\ S - S = 123$ nearly.

$\therefore S = \frac{123}{999} = \frac{41}{333}$ nearly.

The

The firſt ſeries of recurring numbers will be the upper part of the required fraction; and the product of the ſame number of nines, as there are figures in the upper part, will be its lower part.

Thus $\frac{9}{9} = 1 = ,9999 \text{ - - - - - - } 9$

If only one figure of the decimal is taken, the error is conſiderable, being $\frac{1}{10}$; if two figures are taken, the error is $\frac{1}{100}$; if three figures are taken, the error is $\frac{1}{1000}$, and ſo on.

$\frac{3}{9} = \frac{1}{3} = ,333 \text{ - - - - } 3$

If one figure only is taken, the error is $\frac{1}{30}$; if two figures are taken, the error is $\frac{1}{300}$; if three figures are taken, the error is $\frac{1}{3000}$, and ſo on.

The common fractions may, from the rules above given, be eaſily reduced to decimals, and again the decimals may be brought back to the common fractions of the ſame value. Thus, one ſhilling is a twentieth part of a pound, or ,05*l.* Two ſhillings are a tenth part of a pound, or ,1*l.* Three ſhillings are three twentieths of a pound, or ,15l. Four ſhillings are one-fifth of a pound, or ,2*l.* And ſo of any number of ſhillings which may be expreſſed in decimal parts of a pound, either by multiplying ,05 into the given number of ſhillings, or by finding a decimal equal to the fraction of a pound which the given number of ſhillings compoſes.

Again,

Again, a penny is the 12th part of a ſhilling, or the 240th part of a pound. Divide, therefore ,05 by 12, and the reſult ,0041666, or the ſame number derived from finding a decimal equal to $\frac{1}{240}$, is the decimal part of a pound correſponding to one penny. In the ſame manner a farthing, or any number of farthings, may be expreſſed in decimals; a foot may be expreſſed in decimal parts of a yard; an ounce in decimal parts of a pound.

What decimal of a pound is 6*s.* 8*d.*? Anſ. ,3333*

What decimal of a pound is 3*s.* 4*d.*? Anſ. ,1666̸

What decimal of a pound is 11*s.* 3*d.*? Anſ. ,5625.

Again, if the decimal of a pound, yard, chaldron, or other meaſure is given, its value in the inferior diviſions of thoſe quantities may be found. Thus, it is required to find the amount of ,98654*l.* in ſhillings, pence, and farthings.

,98654*l.* = ,98654 × 20 ſhillings = 19,73080 = 19 ſhillings, and ,7308 S.

But ,7308 S = ,7308 × 12 pence = 8,7696 = eight pence, and ,7696 pence.

But ,7696 pence = ,7696 × 4 farthings = 3.0784 = three farthings, and ,0784 decimal parts of a farthing.

∴ ,98654*l.* = 19*s.* 8¾*d* ,0784.

* The mark drawn through the three ſhews, that it may be continued without end.

How

How many ſhillings, pence and farthings are there in ,79648*l.*? Anſ. 15*s.* 11*d.* 0,6208.

How many ſhillings, pence and farthings are there in ,989769*l.*? Anſ. 19*s.* 9½*d.* ,17824.

How many feet and inches are there in ,89563 yards? Anſ. 2*f.* 8*i.* ,24268.

Much trouble would be ſaved in common life, if decimal diviſions were uſed both for money, weights and meaſures, as fractional parts might then be managed as eaſily as whole numbers. If the ſhilling were the tenth part of a pound, the penny the tenth part of a ſhilling, the farthing the tenth part of a penny, a ſum of money would be written down, with only a comma to diſtinguiſh between the pounds and its parts. Thus, 23 pounds, 9 ſhillings, and 8 pence ſix farthings would be thus written down 23,986*l.*

The firſt decimal place after the comma would be that of the ſhillings; the ſecond that of the pence; the third that of the farthings.

In the ſame manner 12 yards, 8 feet, 9 inches would be written down 12,89. A ſum in the rule of three would be thus performed. What is the price of two yards ſeven feet of cloth, when three yards, nine feet, ſix inches coſt 20 pounds 9 ſhillings and eightpence farthing?

3,96 : 20,981 :: 2,7 : *

2,7

146867

41962

3,96) 56,6487 (14,530

396

1704

1584

1208

1188

207

* = 14 pounds, five ſhillings, and three pence.

In thus uſing decimal arithmetick, care only is neceſſary to place the numbers properly on both ſides of the comma. As in common arithmetick, whole numbers are written down by aſſigning the firſt place to units, the ſecond to units multiplied into ten, the third to units multiplied into 10^2, and ſo on; by placing a comma to the right of the units, and writing down numbers to the right of the comma, the firſt place belongs to units divided by ten, the ſecond place to units divided by 10^2, the third place to units divided by 10^3, and ſo on. Thus, 321,321 means $3 \times 10^2 + 2 \times 10 + 1 + \frac{3}{10} + \frac{2}{10^2} + \frac{1}{10^3}$, or three hundred and twenty-one, with three hundred and twenty-one thouſandth parts.

ON THE POWERS AND ROOTS OF NUMBERS.

WHEN a number is multiplied into itſelf, the product is called the power of that number. a^2 is the ſecond power of a; a^3 the third power, and ſo on; $\overline{a+x}^n$ the nth power of $a + x$.

The indexes, it has been before obſerved, are placed over the terms, to ſhew the number of times the letters are found, with the mark of multiplication expreſſed or underſtood between them, in that term. Theſe indexes are thus uſed for the ſake of convenience; and the multiplication or diviſion of powers is performed, by adding or ſubtracting theſe indexes.

$$a^m \times a^n = a^{m+n}.$$

For a^m is the ſame as a repeated m times; a^n is the ſame as a repeated n times; therefore $a^m \times a^n$ would be written down by repeating a, $m + n$ times together; or, more conciſely, by writing it a^{m+n}.

$$a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}$$

$$\frac{a^m}{a^n} = a^{m-n} \text{ or } \frac{1}{a^{n-m}},$$

according as m is greater or leſs than n. For, if a is repeated m times in the upper part of a fraction, and n times in the lower part, by ſtriking out the ſame number of a's in the upper

upper part and lower part, there will be in the upper part, if m is greater than n, a number of a's equal to the difference of m and n, and if n is greater than m, there will be unit in the upper part and a number of a's, in the lower, equal to the difference of n and m.

$$\frac{a^5}{a^3} = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}$$

$$\frac{a^3}{a^5} = \frac{aaa}{aaaaa} = \frac{1}{aa} = \frac{1}{a^2} = \frac{1}{a^{5-3}}.$$

As the different powers of a number are multiplied together by adding their indexes, if the ſame power of a number is repeated ſeveral times in a term, by multiplying its index into the number of times, in which it is found in that term, it is brought into a ſimpler form Thus, $a^m \times a^m = a^{2m}$, $a^m \times a^m \times a^m = a^{3m}$, $a^m \times a^m \times a^m \times \text{---} a^m$ (a being repeated m times) $= a^{nm}$. Let $a^m = x$. $\therefore a^m \times a^m = x^2 =$ ſecond power of $x =$ ſecond power of a^m. $a^m \times a^m \times a^m \text{----} a^m = x \times x \times x \text{---} x = x^n = n$th power of $x = n$th power of a^m. $\therefore a^{nm} =$ nth power of a^m. Hence powers of any number may be raiſed to higher powers, by multiplying the index of the given power into the index of the required power. Find the third power of a^2. $2 \times 3 = 6$; $\therefore$ 3d power of $a^2 = a^6$. The 5th power of $\overline{a+x}^3 = \overline{a+x}^{15}$. The 7th power of $a^2bc^3 = a^{14}b^7c^{21}$.

A compound term, raiſed to any power, may either be written down with the proper index over it, or the product, ariſing from the actual multiplication of the term into itſelf, agreeably to the number of times expreſſed by the index, may be written down. Thus $\overline{a+b}^2$ is the ſecond

power of $a + b$, which is equal to $\overline{a+b} \times \overline{a+b} = a^2 + 2ab + b^2$; and by multiplying a two-fold term into itſelf a ſmall number of times, the reſult of a ſimilar multiplication, for any number of times, may be diſcovered.

$a + b$
$a + b$
$a^2 + ab$
$\quad + ab + b^2$
$a^2 + 2ab + b^2 =$ 2d power.
$a + b$
$a^3 + 2a^2b + ab^2$
$\quad + a^2b + 2ab^2 + b^3$
$a^3 + 3a^2b + 3ab^2 + b^3 =$ 3d power.
$a + b$
$a^4 + 3a^3b + 3a^2b^2 + ab^3$
$\quad + a^3b + 3a^2b^2 + 3ab^3 + b^4$
$a4 + 4a^3b + 6a^2b^2 + 4ab^3 + b4 =$ 4th power.
$a + b$
$a5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4$
$\quad + a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5$
$a5 + 5a^4b + 10.a^3b^2 + 10a^2b^3 + 5ab^4 + b5 =$
$a + b$ 5th power.
$a^6 + 5a^5b + 10a^4b^2 + 10a^3b^3 + 5a^2b^4 + ab5$
$\quad + a^5b + 5a^4b^2 + 10a^3b^3 + 10a^2b^4 + 5ab5 + b^6$
$a^6+6a5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab5+b^6 =$
$a + b$ 6th power.

a7

$$a^7 + 6a^6b + 15a^5b^2 + 20a^4b^3 + 15a^3b^4 + 6a^2b^5 + ab^6$$
$$+ a^6b + 6a^5b^2 + 15a^4b^3 + 20a^3b^4 + 15a^2b^5 + 6ab^6 + b^7$$

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 =$$
7th power

From this multiplication it appears that the literal co-parts of the ſimple terms in each power follow a certain order. The firſt term of the ſecond power is a^2, of the third power a^3, of the fifth power a^5. The ſucceſſive terms have each two literal co-parts; and, ſuppoſing n to repreſent the index of the power to which the twofold term is raiſed, one literal co-part in the ſecond term is a^{n-1}, or a raiſed to a power leſs than the power to which the whole was to be raiſed by unity; the other literal part is b, or the ſecond term in the twofold term. The third term has a raiſed to the power $n - 2$ and b raiſed to the ſecond power, and ſo on. The power of a is diminiſhed by 1 in each ſucceſſive term, b appears in the ſecond term, its ſecond power in the third, and in the ſucceſſive terms its index is increaſed by unity. Each term, after the firſt, has alſo its figure co-part; and if we ſuppoſe A, B, C, D, to repreſent the figure co-parts in each term, $a + b$ raiſed to the nth power may be thus written down:

$$a^n + Aa^{n-1}b + Ba^{n-2}b^2 + Ca^{n-3}b^3 \text{ - - - -}$$
$$\text{- - - - - - } + La^2b^{n-2} + Mab^{n-1} + b^n.$$

The index of a decreaſes by unity, and the index of b increaſes by unity, ſo that if we read from b^n to a^n, the index of b decreaſes in every term in ſucceſſion, in the ſame manner as the index of a decreaſed in reading from a^n to

b^n. The number of terms in any power of $a + b$ is equal to $n + 1$.

Each co-part A, B, C, D is equal to the product of the indexes of a in the preceding terms, divided by the product of the indexes of b in its own and the preceding terms. Thus, in the second power of $a + b$, A = 2 = index of a^2. B = 1 = index of a^2 divided by index of b^2.

In the third power A = 3 = index of a^3. B = 3 = product of the indexes of a^3 and a^2 divided by index of $b^2 = 3 \times 2 \div 2 = 3$. C = 1 = product of indexes of a^3 and a^2, divided by the product of the indexes of b^2 and $b^3 = \frac{3 \cdot 2}{2 \cdot 3} = 1$.

In the 4th power A = 4 = index of a^4. B = 6 = product of indexes of a^4, a^3 divided by the index of $b^2 = \frac{4 \cdot 3}{2} = 6$. C = 4 = product of indexes of a^4, a^3, a^2 divided by the product of indexes of $b^2 b^3 = \frac{4 \cdot 3 \cdot 2}{2 \cdot 3} = 4$. D = product of indexes of a^4, a^3, a^2 divided by the product of the indexes b^2, b^3, $b^4 = \frac{4 \cdot 3 \cdot 2}{2 \cdot 3 \cdot 4} = 1$.

If therefore $a + b$ is to be raised to the nth power, the literal and figure co-parts may be written down without the trouble of a continued multiplication.

$$\overline{a + b}^n = a^n + Aa^{n-1}b + Ba^{n-2}b^2 + Ca^{n-3}b^3 + Da^{n-4}b^4 \\ - \quad - \quad - \quad Mab^{n-1} + b^n.$$

By the above rule A = n.

B = n

$$B = n \cdot \frac{n-1}{2} \quad C = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$$

$$D = n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}.$$

$$\therefore \overline{a+b}^n = a^n + na^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$$

$$a^{n-3}b^3 + \text{ - - - - - - - } nab^{n-1} + b^n.$$

The figure part of any term may be written down, if the preceding term only is known, by multiplying its co-part into the index of a in that term, and dividing the product by the index of b in the required term. Thus $n \cdot \frac{n-1}{2}$ $a^{n-2}b^2$ being the preceding term, the figure co-part in the next term, $a^{n-3}b^3$, is equal to $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$.

If the two-fold term, to be raiſed to any power, conſiſted of two terms, one of which is to be taken from the other, as $a-b$, the mark of taking away will appear alternately in the required power. $a^n - Aa^{n-1}b + Ba^{n-2}b^2 - Ca^{n-3}b^3$.

A threefold term is raiſed to any power, by making two of its terms equal to a ſimple term, and proceeding with the two-fold term thus made by the preceding rule. Let $a+b+c$ repreſent any three-fold term, then if $b+c=d$, $a+b+c = a+d$, and $\overline{a+b+c}^n = \overline{a+d}^n =$

$$a^n + na^{n-1}d + n \cdot \frac{n-1}{2} a^{n-2}d^2 \text{ - - - } nad^{n-1} + d^n$$

$$= a^n + na^{n-1}\overline{b+c} + n \cdot \frac{n-1}{2} a^{n-2}\overline{b+c}^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}$$

$$a^{n-3}\overline{b+c}^3 \text{ - - - } na\overline{.b+c}^{n-1} + \overline{b+c}^n.$$

Let $n = 2$.

$$\overline{a+b+c}^2 = a^2 + 2a \cdot \overline{b+c} + \overline{b+c}^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2.$$

$\overline{a+b+c}^3 = a^3 + 3a^2.\overline{b+c} + 3a.\overline{b+c}^2 + \overline{b+c}^3 = a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc + 3ac^2 + b^3 + 3bc^2 + 3b^2c + c^3$.

In the ſame manner a four-fold term may be raiſed to any power, by making it into a two-fold term; and for every three-fold term that occurs, may, in the ſame manner, be ſubſtituted a two-fold term, till at laſt the ſimple terms in the four-fold term will alone appear with the figure co-parts in the power to which it was to be raiſed.

To find the roots of any number, a number muſt be found, which, multiplied into itſelf once, or oftener if neceſſary, produces the given number. In ſimple terms, when the root required has the ſame index with the power of the number or numbers in the term, the root is the number itſelf under the indexes. Thus the ſecond root of a^2 is a; the third root of b^3 is b; the fourth root of $a^4b^4c^4$ is $a.b.c$; the nth root of $\overline{a+b}^n$ is $a+b$.

$$\text{Let } a^m = b \quad \therefore a^{mn} = b^n.$$

$\therefore$ nth root of b^n is equal to the nth root of a^{mn}; but the nth root of b^n is b, or a^m.

$$\therefore \text{ the } n\text{th root of } a^{mn} \text{ is } a^m, \text{ or } a^{\frac{mn}{n}}.$$

Hence, to find the root of a power whoſe index is not the ſame as the index of the root required, divide the index of the power by the index of the root required. Thus, the 2d root of a^2 is $a^{\frac{2}{2}}$ or a; the third root of a^6 is $a^{\frac{6}{3}}$ or a^2; the fourth root of a^{12} is $a^{\frac{12}{4}}$ or a^3; the 5th root of $\overline{a+b}^{15}$ is $\overline{a+b}^3$.

The

The index of the power to be thus divided may be incapable of diviſion, and then the root required is a power with a fractional index. Thus, the third root of a^2 is $a^{\frac{2}{3}}$. In figures, this root can be found; thus, the third root of 8^2 is $8^{\frac{2}{3}}$; or by writing down the ſecond power of $8 = 64$, and finding the third root of 64, which is 4, we have $4 = 8^{\frac{2}{3}}$. The 5th root of a^4 is $a^{\frac{4}{5}}$. The 8th root of a^{11} is $a^{\frac{11}{8}}$. Thus alſo the different roots of a may be written down with fractional indexes. The ſecond root of a is $a^{\frac{1}{2}}$, the third root $a^{\frac{1}{3}}$, the fourth root $a^{\frac{1}{4}}$. The nth root $a^{\frac{1}{n}}$.

Roots of the ſame number may be multiplied or divided by adding or ſubtracting their indexes.

$$a^{\frac{1}{3}} \times a^{\frac{1}{4}} = a^{\frac{1}{3}+\frac{1}{4}} = a^{\frac{7}{12}}$$

For $\frac{1}{3} = \frac{4}{12}$; $\frac{1}{4} = \frac{3}{12}$

$$\therefore a^{\frac{1}{3}} \times a^{\frac{1}{4}} = a^{\frac{4}{12}} \times a^{\frac{3}{12}} = \sqrt[12]{a^4 \times a^3} = \sqrt[12]{a^7} = a^{\frac{7}{12}} = a^{\frac{1}{3}+\frac{1}{4}}.$$

$$a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{1}{m}+\frac{1}{n}} = a^{\frac{n+m}{nm}}$$

for $\frac{1}{m} = \frac{n}{mn}$, and $\frac{1}{n} = \frac{m}{mn}$

$$\therefore a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{n}{mn}} \times a^{\frac{m}{mn}} = \sqrt[mn]{a^n \times a^m} = \sqrt[mn]{a^{n+m}} = a^{\frac{n+m}{nm}}.$$

In the ſame manner $a^{\frac{m}{n}} \times a^{\frac{p}{q}}$ is proved equal to $a^{\frac{m}{n}+\frac{p}{q}}$.

For

For $\frac{m}{n} = \frac{mq}{nq}$, and $\frac{p}{q} = \frac{np}{nq}$

$\therefore a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \times a^{\frac{np}{nq}} = \sqrt[nq]{a^{mq} \times a^{np}} = \sqrt[nq]{a^{mq+np}}$

$= a^{\frac{mq+np}{nq}} = a^{\frac{m}{n}+\frac{p}{p}}$.

Let $a^{\frac{1}{3}}$ be divided by $a^{\frac{1}{4}}$.

$\frac{1}{3} = \frac{4}{12}$ and $\frac{1}{4} = \frac{3}{12}$

$a^{\frac{1}{3}} \div a^{\frac{1}{4}} = \frac{a^{\frac{4}{12}}}{a^{\frac{3}{12}}} = \sqrt[12]{\frac{a4}{a^3}} = \sqrt[12]{a^{4-3}} = \sqrt[12]{a} = a^{\frac{1}{12}}$.

$a^{\frac{1}{m}}$ divided by $a^{\frac{1}{n}}$ is equal to $a^{\frac{1}{m}-\frac{1}{n}} = a^{\frac{n-m}{mn}}$

For $\frac{1}{m} = \frac{m}{mn}$ and $\frac{1}{n} = \frac{m}{mn}$

$\therefore \frac{a^{\frac{1}{m}}}{a^{\frac{1}{n}}} = \frac{a^{\frac{n}{mn}}}{a^{\frac{m}{mn}}} = \sqrt[mn]{\frac{a^n}{a^m}} = \sqrt[mn]{a^{n-m}}$ or $\sqrt[mn]{\frac{1}{a^{m-n}}}$

But $\sqrt[mn]{a^{n-m}} = a^{\frac{n-m}{mn}}$

and $\sqrt[mn]{\frac{1}{a^{m-n}}} = \frac{1}{a^{\frac{m-n}{mn}}}$

Multiply $a^{\frac{3}{4}} \times a^{\frac{5}{2}} \times a^{\frac{1}{3}}$. Anf. $a^{\frac{43}{12}}$.

Multiply $a^{\frac{1}{3}} b^{\frac{1}{2}} \times a^{\frac{2}{3}} b^{\frac{3}{4}}$. Anf. $a^{\frac{11}{13}} b^{\frac{5}{4}}$.

Multiply

Multiply $\overline{a+b}|^{\frac{3}{2}} \times \overline{a+b}|^{\frac{5}{4}}$. Anſ. $\overline{a+b}|^{\frac{11}{4}}$.

Divide $a^{\frac{3}{7}}$ by $a^{\frac{1}{4}}$. Anſ. $a^{\frac{5}{28}}$.

Divide $a^{\frac{2}{3}}b$ by $a^{\frac{3}{4}}b^{\frac{1}{2}}$. Anſ. $\frac{b^{\frac{1}{2}}}{a^{\frac{1}{12}}}$.

Divide $a^{\frac{3}{5}} \times \overline{a+b}|^{\frac{2}{4}}$ by $b^{\frac{3}{4}} \times \overline{a+b}|^{\frac{5}{2}}$. Anſ. $\frac{a^{\frac{3}{5}}}{b^{\frac{3}{4}}\overline{a+b}|^{\frac{7}{4}}}$.

The roots of compound terms may be frequently found by certain diviſions, diſcovered from obſerving the reſult of the multiplication of two-fold, three-fold, or manifold terms into each other.

Mult $a + b$
into $a + b$

Prod. $a^2 + 2ab + b^2 =$ 2d power of $a + b$.

The firſt term is the ſecond power of a; the ſum of the two next terms is $\overline{2a+b} \times b$. If therefore, in a compound term, the firſt term ſhould contain the ſecond power of a number, and, on taking away this ſecond power, the remainder ſhould be diviſible without a remainder, by a compound term made up of twice the root of the firſt term added to another term, this root, added to the term, will be the ſecond root of the compound term. For thus from the compound term $a^2 + 2ab + b^2$ we gain $a + b$, which, multiplied into itſelf, gives the compound term.

Let

Let it be required to find the ſecond root of 144. Place a dot over the units and hundreds. Find the root of the figure in the place of the hundreds, and multiply this root by 2 and add to it a number, ſuch, that the two next figures may be diviſible by this ſum. The root and the number thus found will be the required root.

$$\begin{array}{rrr} & & \dot{1}44\,(12 \\ & a^2 = & 1\,\ldots \\ \hline 10 \times 2 = & 20 & 22)\;44 \\ \text{Add} & 2 & \quad 44 \\ \hline \text{Sum} & 22 & \quad 00. \end{array}$$

144 is the ſame as $100 + 40 + 4$; the ſecond root of 100 is 10. Multiply 10×2, and divide $40 + 4$ by $20 + 2$, which is an exact divider of it; $\therefore 10 + 2$ is the root of 144.

$$\begin{array}{ll} \text{Multiply} & 10 + 2 \\ \text{into} & 10 + 2 \\ \hline & 100 + 20 \\ & \quad + 20 + 4 \\ \hline \text{Product} & 100 + 40 + 4 = \overline{10 + 2}^2 \end{array}$$

In finding the root of 1, the firſt figure in 144, we found the root of 100, for 1 in that place ſignified 100, and the number in the quotient ſignified 10, as it was to be followed by another number.

The ſecond root of 100 is ten, of 10000 a hundred, of 1000000

1000000 a thouſand, and ſo on. The root therefore of a number leſs than 100 is leſs than 10, and conſiſts of only one figure; the root of a number between a hundred and ten thouſand is more than 10, and leſs than 100, and conſequently it conſiſts of two figures. The root of a number between a thouſand and a million is more than 100, and leſs than 1000, and conſequently conſiſts of three figures, and ſo on. And hence there will be as many figures in the root as there are points placed over the number whoſe root was to be found.

Find the ſecond root of 576.

$$\begin{array}{rrl} & & 576 \,(\, 24 \\ & a^2 = & 4\,. \\ \hline 2a = & 40 & 44\,)\,176\,(\,4 \\ +\,b^2 = & 4 & 176 \\ \hline & 44 & 000. \end{array}$$

$$576 = 20^2 + 2 \cdot 20 \cdot 4 + 4^2$$

And the root is $20 + 4$, or 24.

The ſecond power of a compound number $a + b + c + d$ is $a^2 + 2ab + 2ac + 2ad + 2bc + \text{ - - - } + b^2 + c^2 + d^2$
$= a^2 + 2ab + b^2 + 2.\overline{a.+b.}c + c^2 + 2.\overline{a+b+c.}\times d + d^2 +$.

To find the root therefore of a number which has more than two points: Find, as before, the root a of the firſt period; and to the remainder annex the ſecond period, which divide by $2ab + b^2$; and to the remainder annex the next period, which divide by $2.\overline{a+b} \times c + c^2$, and ſo on.

Find

Find the ſecond root of 15376.

$$\begin{array}{rl} & \dot{1}\dot{5}\dot{3}\dot{7}\dot{6} \;(\; 124 \\ a^2 = & 1\ldots \\ \hline & 053\ldots \\ 2ab = 2.10..2 \\ b^2 = \quad 4.. & \Big\} = \quad 44\ldots \\ \hline & 976 \\ 2.\overline{a+b}.c = 2.120.4 & \Big\} = \quad 976 \\ +\,c^2 = \quad 16 & \\ \hline & 000. \end{array}$$

$15376 = 100^2 + 2.100.20 + 20^2 + 2.120.4 + 4^2 = 10000 + 4000 + 400 + 960 + 16.$

The root of 10000 is $100 = a$. The 53 brought down is equal to 4400 + 900; and the $4400. = 2.100.20 + 20^2 = 2ab + b^2$: there remains therefore 976, which muſt be equal to $2.\overline{a+b}.c + c^2 = 2.120.4 + 4^2 = 960 + 16 = 976.$

124 is therefore the root of 15376.

The roots of any numbers, repreſented by common figures, may thus be found; for the number being marked off by points over the places of units, hundreds, ten thouſands, and ſo on, will repreſent a compound term, whoſe parts are ſimilar to the terms $a^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cb + b^2 + c^2 + d^2$, with more or leſs products, according to the number of points before that on the place of units.

in finding the ſecond root of a number, there ſhould emainder left after the laſt diviſion, the operation e continued by the uſe of decimals.

Find

Find the ſecond root of 112.

$$\begin{array}{rl} & 112\,)\,10{,}58 \\ a^2 = & 1\,0\,0 \\ 2a\,.\,b = 2\,.\,10\,.\,0 & \\ +\,b^2 \quad +\,0^2 & 012. \end{array}$$

$$\left.\begin{array}{rl} 2\,.\,\overline{a+b}\,.\,c = 2\,.\,10\,.\,\times,5 & \\ +\,c^2 = \quad ,25 & \end{array}\right\} \begin{array}{r} 12{,}00 \\ = 10{,}25 \end{array}$$

$$\left.\begin{array}{rl} 2\,.\,\overline{a+b+c}\,.\,d = 2\,.\,10{,}5\,.\,08 & \\ +\,c^2 \quad = \quad ,0064 & \end{array}\right\} \begin{array}{r} 1{,}7500 \\ = 1{,}6864 \\ \hline 636. \end{array}$$

The root of 100 is 10, therefore $a = 10$, and the remainder is 12: conſequently this remainder is not diviſible by twice the firſt figure in the root. Therefore the ſecond figure or b, in this caſe, equals 0, and two cyphers are annexed to the remainder, that a number c may be found, which, multiplied into $2a + b$, may be either equal or nearly equal to 12; this number is ,5: and, by continuing in the ſame manner, other decimal figures correſponding to d, e, f are found; and the longer the operation is continued, the nearer is the approach made to the ſecond root of the given number.

Find the ſecond root of 4567 to two places of decimals.

Find the ſecond root of 82 to three places of decimals.

Find the ſecond root of 94 to four places of decimals.

Find

Find the ſecond root of 333 to five places of decimals.

The third root of a number is found in a ſimilar manner. The number is made to correſpond with the term $a^3 + 3a^2b + 3ab^2 + b^3 + 3 \cdot \overline{a+b}^2 \cdot c + 3\overline{a+b} \cdot c^2 + c^3 + 3 \cdot \overline{a+b+c}^2 \cdot d + 3\overline{a+b+c} \cdot d^2 + d^3$ - - - - which is the third power of any compound term $a + b + c + d$ - - - - - . As, in finding the ſecond root, periods were made by pointing every ſecond place from the units, in this caſe every third place from the units is pointed off. For the third root of 1000 is 10, and conſequently the third root of a number leſs than 1000 is leſs than 10, and conſiſts of only one figure; the third root of 1000000 is 100, and conſequently the third root of any number between 10000 and 1000000 is more than 10, and leſs than 100, that is, it conſiſts of two figures; and ſo on of the reſt.

Find the third root of 10648.

10648 (22

$2^3 \ldots = 8 \ldots = a^3$

$3 \cdot a^2 \cdot b = 3 \cdot 20^2 \cdot 2 = 2400$
$3ab^2 = 3 \cdot 20 \cdot 2^2 = 240$
$b^3 = 2^3 = 8$

2648
2648
0000.

2, the third root of 8, is the leaſt number whoſe third root is neareſt to 10, and therefore 20 is the neareſt third root to be found in 10000. Conſequently, 20 being made equal to a, the remainder 2648 muſt be equal to $3a^2b + 3ab^2 + b^3$; and the neareſt number multiplied into $3a^2$ which makes the neareſt divider of

2648 is 2, for $3a^2b = 2400$, and the remainder 648 is equal to $3ab^2 + b^3$, and consequently $a + b$ or $20 + 2$ is the third root of the propoſed number.

Mult.	22	
into	22	
	44	
	44	
Mult.	484	ſecond power
into	22	
	968	
	968	
	10648	= third power.

Find the third root of 44361864.

$$\begin{array}{r} 44361864 \,)\, 354 \\ a^3 = 27 \\ \hline 17361 \\ 3a^2b = 3 \cdot 30^2 \cdot 5 = 13500 \\ 3ab^2 = 3 \cdot 30 \cdot 5^2 = 2250 \\ b^3 = 5^3 = 125 \\ \hline = 15875 \\ \hline 1486864 \\ 3\cdot\overline{a+b}^2 \cdot c = 3\cdot\overline{350}^2 \cdot 4 = 1470000 \\ 3\cdot a+b\cdot c^2 = 3\cdot 350\cdot 4^2 = 16800 \\ c^3 = 4^3 = 64 \\ \hline 1486864 \\ \hline 0000000 \end{array}$$

354 is the root.

Frequently in performing this operation the learner is put to conſiderable difficulty in finding the ſecond figure

b, for he may take ſuch a number as ſhall at firſt ſight enable him to take away $3a^2b + 3ab^2 + b^3$ from the firſt remainder, but on examination he finds it too great. Thus, if he attempts to divide the firſt remainder 17361 by $3a^2 \times 6$, or $3 . \overline{30}^2 . 6 = 16200$, he finds it capable of diviſion; but on proceeding to add $3ab^2 + b^3$, he will find the ſum greater than the firſt remainder, and conſequently 6 cannot be the ſecond figure of the root. He therefore tries 5, which ſucceeds. He muſt find ſuch a number for his ſecond figure, which, multiplied into $3a^2$, gives not only a product that may be taken from the firſt remainder, but leaves alſo a remainder diviſible by $3ab^2 + b^3$, and ſo of the other remainders.

Find the third root of 13.

$$
\begin{array}{rl}
 & 13\ (\ 2{,}35\ \text{- - - - -} \\
a^3 = 8 & \\
\hline
3a^2b^2 = 3 \cdot 2^2 \cdot {,}3 = {,}36 & \\
3ab^2 = 3 \cdot 2 \cdot {,}3^2 = {,}54 &)\ 5{,}000 \\
b^3 = \quad\ {}^2 \quad\ {,}027 & \ \ 4{,}167 \\
\hline
3 \cdot \overline{a+b} \cdot c = 3 \cdot 2{,}3 \cdot {,}05 = {,}7935 & \\
3 \cdot \overline{a+b} \cdot c^2 = 3 \cdot 2{,}3 \cdot \overline{,05}^2 = {,}01725 & \ \ {,}833000 \\
c^3 = \quad\ \overline{,05}^3 = {,}000125 & \ \ 810875 \\
\hline
 & {,}22125000.
\end{array}
$$

Find the third root of 11 to three places of decimals.

Find the third root of 126 to two places of decimals.

Find the third root 98,7 to two places of decimals.

The

The anſwer is not given to theſe queſtions, as the learner may ſee whether he has done right by raiſing the root found to the third power.

In the ſame manner as the ſecond and third roots have been found, any other root may be found. Let it be required, for example, to find the nth root of $a^n + na^{n-1}b + n.\frac{n-1}{2}a^{n-2}b^2 + n.\frac{n-1}{2}.\frac{n-2}{3}a^{n-3}b^3 - - nab^{n-1} + b^n$.

Find the nth root of the firſt term which is a, and raiſe a to the $\overline{n-1}^{th}$ power, and multiply it, thus raiſed, into n, and find a number, which, multiplied into this product, is a divider of the ſecond term; and if this number b is of ſuch a value that $n^{-1}b + n.\frac{n-1}{2}a^{n-2}b^2$ - - - - - - being taken away from the firſt remainder leaves no remainder, the root is a two-fold term; if a remainder is left, then $\overline{a+b}^{th}$ muſt be raiſed to the $\overline{n-1}^{th}$ term, and multiplied by n; from thence to find, as before, the third term c, which is to be ſuch, that $n.\overline{a+b}^{n-1}.c + n.\frac{n-1}{2}\overline{a+b}^{n-2}$ c&c. - - - - - - may be taken from the remainder; and if now no remainder is left, the root is a three-fold term; if a remainder is left, the operation in a ſimilar manner.

In figures, for the fourth root, the pointing is made on every fourth figure from the place of units; for the fifth root on the fifth place, and ſo on for the nth root on the nth place.

ON EQUATIONS OF THE SECOND ORDER.

AN equation having, in any term, an unknown quantity of the ſecond power, and no higher power in any other term, is called an equation of the ſecond order, and all equations of this order are reduceable by the preceding rules to the following forms.

1. $x^2 = b$
2. $x^2 + ax = b$
3. $x^2 - ax = b$
4. $ax - x^2 = b$.

1. Let $c + \frac{x^2}{d} = ex^2 - f$ be an equation to be reſolved. Proceed with it as with a ſimple equation, bringing all the known quantities to the one ſide, and the unknown to the other. Then

$$c + f = ex^2 - \frac{x^2}{d} = \frac{dex^2 - x^2}{d} = x^2 \times \frac{de - 1}{d}.$$

Divide both ſides of the equation by $\frac{de - 1}{d}$.

$$\therefore \frac{d}{de - 1} \times \overline{c + f} = x^2.$$

This equation is of the ſecond order of the firſt form; $\frac{d}{de - 1} \times \overline{c + f}$ being in this caſe equal to b in that form.

2. Let

2. Let $cx^2 - d = e - dx^2 - \frac{x}{e}$.

$$\therefore cx^2 + dx^2 + \frac{x}{e} = e + d.$$

$$\therefore x^2 \times \overline{c + d} + \frac{x}{e} = e + d.$$

Divide both sides by $c + d$.

$$\therefore x^2 + \frac{x}{e \cdot \overline{c + d}} = \frac{e + d}{c + d}$$

This equation is therefore of the second order and of the second form, $x^2 + ax = b$; for a may be equal to $\frac{1}{e \cdot \overline{c + d}}$ and b equal to $\frac{e + d}{c + d}$.

3. Let $d - cx^2 + ex = f - \frac{x}{d}$.

then $d - f = cx^2 - ex - \frac{x}{d} = cx^2 - x \times \frac{e + 1}{d}$.

Divide both sides by c.

$$\therefore \frac{d - f}{c} = x^2 - x \times \frac{e + 1}{cd}.$$

Hence this equation is of the second order and of the third form, $x^2 - ax = b$, for $\frac{e + 1}{cd}$ corresponds with a, and $\frac{d - f}{c}$ corresponds with b.

4. Let $c = dx + \frac{bx^2}{e} = fx - x^2 + d$:

then $c - d = fc + dx - \frac{bx^2}{e} - x^2 = x \times \overline{f + d} - x^2 \times \frac{b + e}{e}$.

Divide both ſides by $\frac{b + e}{e}$.

$$\therefore e \times \frac{c - d}{b + e} = e.x \times \frac{f + d}{b + e} - x^2.$$

This equation is of the ſecond order and of the fourth form $ax - x^2 = b$; for $e \times \frac{f + d}{b + e}$ correſponds with a, and $e \times \frac{c - d}{b + e}$ correſponds with b.

If in a propoſed equation the unknown number appears in the lower part of a term, the equation is not to be eſteemed of the ſecond order, though in one of the terms the unknown number is raiſed to the ſecond power, becauſe on multiplying both ſides of the equation by the unknown number, by which it ceaſes to appear in the lower part of a term, the equation will have a higher power than the ſecond of the unknown number. $\frac{a}{x} + bx^2 = cx + d$ is not an equation of the ſecond order; for to bring it to one of the forms laid down, the known numbers muſt be brought to one ſide, and the unknown to the other, and then

$$\frac{a}{x} + bx^2 - cx = d.$$

Multiply

Multiply both ſides by x.

$$\therefore a + bx^3 - cx^2 = dx$$

$$\therefore a = dx - bx^3 + cx^2.$$

This equation has the third power of x in one of its terms, and is therefore not of the ſecond order.

To reſolve an equation in the firſt form, the roots of each ſide of the equation are to be found; and as on one ſide the root is x, and on the other ſide there are only known quantities, the value of x, after this proceſs, will be given.

Let $5x^2 = 45$.

$$\therefore x^2 = \frac{45}{5} = 9$$

$$\therefore \sqrt{x^2} = \sqrt{9}$$

$$\therefore x = 3.$$

Let $34 + x^2 = 5x^2 - 30$.

$$\therefore 34 + 30 = 5x^2 - x^2$$

$$\therefore 4x^2 = 64$$

$$\therefore x^2 = 16$$

$$\therefore \sqrt{x^2} = \sqrt{16}$$

$$\therefore x = 4.$$

Let $\frac{5x^2}{8} - 13 = 12 - \frac{7x^2}{4}$.

$$\therefore \frac{5x^2}{8} + \frac{7x^2}{4} = 12 + 13.$$

$$\therefore \frac{20x^2 + 56x^2}{32} = 25$$

$$\therefore x^2 \times \frac{76}{32} = 25$$

$$\therefore x^2 = 25 \times \frac{32}{76} = 25 \times \frac{16}{38}$$

$$\therefore \sqrt{x^2} = \sqrt{\frac{25 \times 16}{38}}$$

$$\therefore x = \frac{5 \times 4}{\sqrt{38}} = \frac{20}{\sqrt{38}}.$$

2. The second form is resolved by adding a term to it, which shall make it correspond with the second power of a double term. Now the two first terms in $y^2 + 2cy + c^2$, the second power of $y + c$, corresponds with one side of the equation in this form, $x^2 + ax = b$; for y^2 corresponds with x^2, and 2c 20 with a: but c^2 is the second power of half the co part of y in the term $2cy$; therefore by adding to $x^2 + ax$ the second power of half the co-part of x in the term ax, a compound term will be formed corresponding entirely to $y^2 + 2cy + c^2$. Half of the co-part of x, in the term ax is $\frac{a}{2}$, and its second power is $\frac{a^2}{4}$, and by adding $\frac{a^2}{4}$ to both sides of the equation, it becomes

$$x^2 + ax + \frac{a^2}{4} = b + \frac{a^2}{4}.$$

Take the root on both sides.

$$\therefore \sqrt{x^2 + ax + \frac{a^2}{4}} = \sqrt{b + \frac{a^2}{4}}$$

The

The root of $x^2 + ax + \frac{a^2}{4}$ is $x + \frac{a}{2}$.

$$\therefore x + \frac{a}{2} = \sqrt{b + \frac{a^2}{4}}$$

$$\therefore x = \sqrt{b + \frac{a^2}{4}} - \frac{a^2}{2}.$$

Hence, to resolve equations in this form, the rule is, Add to both sides the second power of half the co-part of the simple unknown number, and then take the roots on both sides. The result will give a simple equation, from which the unknown quantity is to be discovered by the preceding rules.

Let $x^2 + 8x = 9$.

8 is the co-part of the unknown number, of which the half is 4 and its square 16.

Add therefore 16 to both sides, and

$$x^2 + 8x + 16 = 9 + 16 = 25$$

$$\therefore \sqrt{x^2 + 8x + 16} = \sqrt{25}$$

$$\therefore x + 4 = 5$$

$$\therefore x = 5 - 4 = 1.$$

Let $x^2 + 10x = 24$.

$$\therefore x^2 + 10x + 25 = 24 + 25$$

$$\therefore \sqrt{x^2 + 10x + 25} = \sqrt{49}$$

$$\therefore x + 5 = 7$$

$$\therefore x = 7 - 5 = 2.$$

3. The

3. The third form is reſolved in a ſimilar manner to the ſecond; for by adding to each ſide of the equation the ſecond power of half the co-part of the ſimple unknown number, the root on one ſide is a double term, conſiſting of the unknown number and half the co-part abovementioned; and the other ſide is the root of known numbers.

$$x^2 - ax = b.$$

Add to both ſides the ſecond power of $\frac{a}{2}$.

$$\therefore x^2 - ax + \frac{a^2}{4} = b + \frac{a^2}{4}$$

$$\therefore \sqrt{x^2 - ax + \frac{a^2}{4}} = \sqrt{b + \frac{a^2}{4}}.$$

Now, of the two double terms which may be the roots of $x^2 - ax + \frac{a^2}{4}$, namely, $x - \frac{a}{2}$, and $\frac{a}{2} - x$, the latter cannot be applied in this caſe; for then $\frac{a}{2}$ would be greater than x, and conſequently a would be much greater than x; and in that caſe ax would be greater than x^2, and ax could not be taken from x^2. But in this form ax is to be taken from x^2, and conſequently $\frac{a}{2}$ cannot be greater than x. Hence $x - \frac{a}{2}$ is the root on one ſide of the equation, and

$$x - \frac{a}{2} = \sqrt{b + \frac{a^2}{4}}$$

$$\therefore x = \sqrt{b + \frac{a^2}{4}} + \frac{a}{2}.$$

Let

Let $x^2 - 4x = 5$.

$$\therefore x^2 - 4x + 4 = 5 + 4 = 9$$
$$\therefore \sqrt{x^2 - 4x + 4} = \sqrt{9}$$
$$\therefore x - 2 = 3$$
$$\therefore x = 3 + 2 = 5.$$

Let $y^2 - 6y = 27$.

$$\therefore y^2 - 6y + 9 = 27 + 9 = 36$$
$$\sqrt{y^2 - 6y + 9} = \sqrt{36}$$
$$\therefore y - 3 = 6$$
$$\therefore y = 6 + 3 = 9.$$

Let $6x - \frac{5x^2}{4} = 3x^2 - 12$.

$$\therefore 3x^2 + \frac{5x^2}{4} - 6x = 12$$
$$\therefore \frac{12x^2 + 5x^2}{4} - 6x = 12$$
$$\therefore \frac{17x^2}{4} - 6x = 12$$
$$\therefore x^2 - \frac{4 \cdot 6 \cdot x}{17} = \frac{12 \cdot 4}{17}$$
$$\therefore x^2 - \frac{24 \cdot x}{17} + \frac{12^2}{17^2} = \frac{12 \cdot 4}{17} + \frac{12^2}{17^2}$$
$$\therefore \sqrt{x^2 - \frac{24x}{17} + \frac{12^2}{17^2}} = \sqrt{12 \cdot \frac{4 \cdot 17 + 12}{17^2}}$$

$\therefore x$

$$\therefore x - \frac{12}{17} = \sqrt{12 \cdot \frac{4 \cdot 17 + 12}{17^2}}$$

$$\therefore x = \sqrt{\frac{12 \cdot 4 \cdot 17 + 12}{17}} + 12 = 2{,}525.$$

$$\text{Let } \frac{6}{x+1} + \frac{2}{x} = 3.$$

$$\therefore 6x + 2x + 2 = 3x^2 + 3x$$

$$\therefore 3x^2 + 3x - 8x = 2$$

$$\therefore 3x^2 - 5x = 2$$

$$\therefore x^2 - \frac{5x}{3} = \frac{2}{3}$$

$$\therefore x^2 - \frac{5x}{3} + \frac{25}{36} = \frac{2}{3} + \frac{25}{36}$$

$$\therefore \sqrt{x^2 - \frac{5x}{3} + \frac{25}{36}} = \sqrt{\frac{2}{3} + \frac{25}{36}} = \sqrt{\frac{24 + 25}{36}}$$

$$\therefore x - \frac{5}{6} = \frac{7}{6}$$

$$\therefore \frac{7}{6} + \frac{5}{6} = 2.$$

4. To reſolve the equations of the fourth form, the mode adopted in the two preceding forms fails; for by adding $\frac{a^2}{4}$ to $ax - x^2$, one ſide of an equation of the fourth form, a compound term $ax - x^2 + \frac{a^2}{4}$ is made, which is not ſimilar to the ſecond power of a two-fold term. Inſtead therefore of adding the ſecond power of half the co-part of the unknown term to each ſide of the equation

equation, each is taken away from $\frac{a^2}{4}$, and one ſide becomes the ſecond power of a two-fold term, and the other ſide has only known numbers.

$$\text{From } \frac{a^2}{4} = \frac{a^2}{4}$$

$$\text{take } ax - x^2 = b.$$

$$\therefore \frac{a^2}{4} - ax + x^2 = \frac{a^2}{4} - b.$$

$$\therefore \sqrt{\frac{a^2}{4} - ax + x^2} = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

The root of a compound term $\frac{a^2}{4} - ax + x^2$ is either $\frac{a}{2} - x$, or $x - \frac{a}{2}$, according as $\frac{a}{2}$ is greater or leſs than x; but in the equation $ax - x^2 = b$, x may be greater or leſs than $\frac{a}{2}$, and yet, in both caſes, the difference between ax and x^2 be equal to ſome number: but x cannot be greater than a, for then x^2 could not be taken away from ax.

$ax - x^2 = x \times \overline{a - x}$; conſequently x muſt be leſs than a, or $a - x$ will not repreſent any number whatever, ſince, if x is greater than a, it cannot be taken from a. But the learner muſt take care, in writing down ſuch an equation, that

that the ſecond power of half the co-part of x is not leſs than b; for then $\sqrt{\frac{a^2}{4} - b}$ would be an abſurdity: b could not then be taken from $\frac{a^2}{4}$, and conſequently $\sqrt{\frac{a^2}{4} - b}$ could not repreſent any number whatever.

$$\sqrt{x^2 - ax + \frac{a^2}{4}} = \sqrt{\frac{a^2}{4} - b}$$

$$\therefore x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b}$$

$$\text{and } \frac{a}{2} - x = \sqrt{\frac{a^2}{4} - b}$$

From the firſt of the two latter equations,

$$x = \sqrt{\frac{a^2}{4} - b} + \frac{a}{2}.$$

From the ſecond,

$$x = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b}.$$

Let $12x - x^2 = 35$.

The ſecond power of half of the co-part of x is 36.

$$\text{From } 36 = 36$$
$$\text{take } 12x - x^2 = 35.$$
$$\therefore 36 - 12x + x^2 = 36 - 35 = 1$$
$$\sqrt{36 - 12x + x^2} = \sqrt{1}$$
$$\therefore 6 - x \text{ or } x - 6 = 1$$

$\therefore x$

$$\therefore x = 1 + 6 = 7$$
$$\text{or } x = 6 - 1 = 5.$$

Both numbers, 7 and 5, ſubſtituted for x in the term $12x - x^2$, give for difference 35.

Let $4x - x^2 = 3$.

$$\therefore 4 - 4x + x^2 = 4 - 3 = 1$$
$$\therefore \sqrt{4 - 4x + x^2} = 1$$
$$\therefore 2 - x \text{ or } x - 2 = 1$$
$$\therefore x = 2 - 1 = 1.$$
$$\text{or } x = 1 + 2 = 3.$$

Let $x + \sqrt{5x + 10} = 8$.

$$\therefore \sqrt{5x + 10} = 8 - x$$
$$\therefore 5x + 10 = 64 - 16x + x^2$$
$$\therefore 5x + 16x - x^2 = 64 - 10$$
$$\therefore 21x - x^2 = 54$$
$$\therefore \frac{441}{4} - 21x + x^2 = \frac{441}{4} - 54 = \frac{441 - 216}{4} = \frac{225}{4}$$
$$\therefore \sqrt{\frac{441}{4} - 21x + x^2} = \sqrt{\frac{225}{4}}$$
$$\frac{21}{2} - x = \frac{15}{2}$$
$$\therefore x = \frac{21}{2} - \frac{15}{2} = \frac{6}{2} = 3.$$

In this caſe the root $x = \frac{21}{2}$ cannot be uſed; for, by the conditions of the queſtion, x is leſs than eight.

SO-

SOLUTION OF PROBLEMS

INVOLVING ONLY ONE UNKNOWN NUMBER, AND PRODUCING EQUATIONS OF THE SECOND ORDER.

IF a queſtion written down in algebraical characters gives an equation of the ſecond order, the unknown number is to be found by the preceding rules; and, if the equation belongs to either of the three firſt forms, there can be only one number to anſwer the conditions of the queſtion. If the equation belongs to the fourth form, there may be either two numbers to anſwer the conditions of the queſtion, or the conditions of the queſtion will ſhew at once which of the two roots $x - \frac{a}{2}$ or $\frac{a}{2} - x$ gives the proper ſolution of the queſtion.

What number is that to whoſe ſecond power if two is added, the ſum is equal to 18 ?

Let $x =$ the number.

Then $x^2 =$ its ſecond power.

$$\therefore x^2 + 2 = 18$$

$$\therefore x^2 = 18 - 2 = 16$$

$$\therefore \sqrt{x^2} = \sqrt{16}$$

$$\therefore x = 4.$$

What number is that, from whoſe ſecond power four being taken away, the remainder is 21 ? Anſ. 5.

What number is that, from three-fourths of whoſe ſecond power twelve being taken away, the remainder is equal to 180?

$$\frac{3x^2}{4} - 12 = 180.$$

$$\therefore \frac{3x^2}{4} = 180 + 12 = 192$$

$$\therefore x^2 = 192 \times \frac{4}{3} = 64 \times 4$$

$$\therefore \sqrt{x^2} = \sqrt{64 \times 4}$$

$$\therefore x = 8 \times 2 = 16.$$

What number is that, the fourth of whoſe ſecond power being taken away from eight, leaves a remainder equal to four? Anſ. 4.

A perſon being aſked what was his age, anſwered, that $\frac{3}{4}$ of his age multiplied by $\frac{1}{12}$ of his age gave a product equal to his age. What was his age?

Let x = his age in years.

$$\therefore \tfrac{3}{4}x \times \tfrac{1}{12}x = x$$

$$\therefore \frac{3x^2}{48} = x$$

$$\therefore 3x = 48$$

$$\therefore x = 16.$$

A perſon bought a certain number of oxen for eighty guineas, and if he had bought four more for the ſame ſum, they would have coſt a guinea a piece leſs. What was the number of oxen, and price of each?

Let x be the number:

then $\frac{80}{x}$ is the price of each;

and $\frac{80}{x+4}$ is the price of each, on ſuppoſition that the perſon had bought four more for 80 guineas.

$$\therefore \frac{80}{x+4} = \frac{80}{x} - 1$$

$$\therefore 80x = 80x + 320 - x^2 - 4x$$

$$\therefore x^2 + 4x + 80x - 80x = 320$$

$$\therefore x^2 + 4x = 320$$

$$\therefore x^2 + 4x + 4 = 320 + 4$$

$$\therefore \sqrt{x^2 + 4x + 4} = \sqrt{324}$$

$$\therefore x + 2 = 18$$

$$\therefore x = 18 - 2 = 16$$

$$\therefore \frac{80}{x} = \frac{80}{16} = 5.$$

$\therefore$ The perſon bought ſixteen oxen at five guineas a piece.

Divide the number 20 into two parts, ſuch, that the product of the whole number into one of the parts ſhall be the ſecond power of the other part.

Let x = the greater part,

$$\therefore 20 - x = \text{the leſs.}$$

$$\therefore x^2 = 20 \times \overline{20 - x} = 400 - 20x$$

$$\therefore x^2 + 20x = 400$$

$$\therefore x^2 + 20x + 100 = 400 + 100$$

$$\therefore \sqrt{x^2 + 20x + 100} = \sqrt{500}$$

$\therefore x$

$$\therefore x + 10 = \sqrt{500}$$

$$\therefore x = \sqrt{500} - 10 = 12{,}36 \text{ nearly:}$$

and the other part $= 7{,}63$ nearly.

Some company, dining together in an inn, find their bill amounts to 175 ſhillings. Two of them were not allowed to pay, and the reſt found that their ſhares amounted to ten ſhillings a man more than if all had paid. Queſ. How many were in company?

Let $x =$ the number.

$\therefore \frac{175}{x} =$ each man's ſhare, if all had paid;

$x - 2 =$ number of thoſe who paid;

and $\frac{175}{x-2} =$ ſhare of each man who paid.

$$\therefore \frac{175}{x-2} - \frac{175}{x} = 10$$

$$\therefore 175 \cdot x - 175x + 350 = 10x^2 - 20x$$

$$\therefore 10x^2 - 20x = 350$$

$$\therefore x^2 - 2x = 35$$

$$\therefore x^2 - 2x + 1 = 35 + 1 = 36$$

$$\therefore \sqrt{x^2 - 2x + 1} = \sqrt{36}$$

$$\therefore x - 1 = 6$$

$\therefore x = 6 + 1 = 7 =$ number of perſons in company:

$x - 2 = 7 - 2 = 5 =$ number of perſons who paid:

$\frac{175}{x-2} = \frac{175}{5} = 35 =$ ſum paid by each of the five perſons.

There are three numbers in geometrical proportion. The ſum of the firſt and ſecond is 10, and the difference of the ſecond and third is 24. What are the numbers?

Let the firſt number $= x$,

$\therefore$ 2d number $= 10 - x$

and 3d number $= 34 - x$

$\therefore x : 10 - x :: 10 - x : 34 - x$

$\therefore x \times \overline{34 - x} = \overline{10 - x}^2$

$\therefore 34x - x^2 = 100 - 20x + x^2$

$\therefore 34x - x^2 + 20x - x^2 = 100$

$\therefore 54x - 2x^2 = 100$

$\therefore 27x - x^2 = 50$

$\therefore \frac{729}{4} - 27x + x^2 = \frac{729}{4} - 50 = \frac{529}{4}$

$\therefore \sqrt{\frac{729}{4} - 27x + x^2} = \sqrt{\frac{529}{4}}$

$\therefore \frac{27}{2} - x = \frac{23}{2}$

$\therefore \frac{27}{2} - \frac{23}{2} = x$

$\therefore x = \frac{4}{2} = 2$

and the numbers are 2, 8, 32.

What are the two numbers whoſe difference is three, and difference of their third powers 117?

Let

Let x = the leaſt number.

$\therefore x + 3$ = greateſt number,

$\therefore \overline{x + 3}^3 - x^3$ = difference of their third powers.

$$\therefore 9x^2 + 27x + 27 = 117$$

$$\therefore 9x^2 + 27x = 117 - 27 = 90$$

$$\therefore x^2 + 3x = 10$$

$$\therefore x^2 + 3x + \frac{9}{4} = 10 + \frac{9}{4} = \frac{49}{4}$$

$$\therefore \sqrt{x^2 + 3x + \frac{9}{4}} = \sqrt{\frac{49}{4}}$$

$$\therefore x + \frac{3}{2} = \frac{7}{2}$$

$$\therefore x = \frac{7}{2} - \frac{3}{2} = 2$$

$$\therefore x + 3 = 2 + 3 = 5.$$

The two numbers therefore are 2 and 5.

What two numbers are thoſe whoſe difference is two, and product of their third powers 42875 ?

Let x and $x + 2$ be the two numbers.

$$\therefore x^3 \times \overline{x + 2}^3 = 42875$$

$$\therefore x \times \overline{x + 2} = \sqrt[3]{42875} = 35$$

$$\therefore x^2 + 2x = 35$$

$$\therefore x^2 + 2x + 1 = 36$$

$$\therefore \sqrt{x^2 + 2x + 1} = \sqrt{36}$$

$$\therefore x + 1 = 6$$

$$\therefore x = 6 - 1 = 5$$

$$\therefore x + 2 = 5 + 2 = 7$$

The two numbers are 5 and 7.

A piece of cloth, which coſt thirty pounds, was ſold for ſuch a ſum, that if the price at which it was ſold is multiplied into the gain, the product is equal to the third power of the gain. What is the gain?

A perſon having bought cloth for 33*l.* 15*s.* ſold it again at 48 ſhillings a piece, and gained as much as one piece coſt. What was the number of pieces?

What two numbers are thoſe whoſe product is 15, and the ſum of their ſecond powers 34?

What number is that, which if you take away from 10, and multiply the remainder by itſelf, the product ſhall be 21?

What two numbers are they whoſe difference is 8 and product 240?

EQUATIONS OF THE SECOND ORDER

FORMED FROM EQUATIONS HAVING UNKNOWN NUMBERS, OF A HIGHER POWER THAN THE SECOND, OR HAVING MORE THAN ONE UNKNOWN NUMBER.

From ſome of the above inſtances, it appears that the queſtion may involve an unknown number of a higher power than the ſecond, and yet the equation reſulting is only of the ſecond order; the unknown number of the third power being made to diſappear upon the prineiples of ſimple equations. Frequently an equation with powers much higher

higher than the ſecond, may be reſolved by bringing them into the forms of the ſecond, which is done by ſubſtituting ſome unknown ſimple number, for the unknown number raiſed to any power. Thus y may be made equal to x^2, and then $y^2 = x^4$: or z may be equal to x^4, and then $z^2 = x^8$.

$$\text{Let } y^4 - 6y^2 = 27$$

$$\text{Make } z = y^2$$

$$\therefore z^2 = y^4$$

$$\therefore z^2 - 6z = 27$$

$$\therefore z^2 - 6z + 9 = 27 + 9 = 36$$

$$\therefore \sqrt{z^2 - 6z + 9} = \sqrt{36}$$

$$\therefore z - 3 = 6$$

$$\therefore z = 9$$

$$\text{But } z = y^2$$

$$\therefore y^2 = 9$$

$$\therefore \sqrt{y^2} = \sqrt{9}$$

$$\therefore y = 3.$$

$$\text{Let } y^6 + ry^3 = \frac{q^3}{27}$$

$$\text{Make } z = y^3$$

$$\therefore z^2 = y^6$$

$$\therefore z^2 + rz = \frac{q^3}{27}$$

$$\therefore z^2 + rz + \frac{r^2}{4} = \frac{q^3}{27} + \frac{r^2}{4}$$

$\therefore$

$$\therefore \sqrt{z^2 + rz + \frac{r^2}{4}} = \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}$$

$$\therefore z + \frac{r}{2} = \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}$$

$$\therefore z = \sqrt{\frac{q^3}{27} + \frac{r^2}{4}} - \frac{r}{2}$$

but $z = y^3$

$$\therefore y^3 = \sqrt{\frac{q^3}{27} + \frac{r^2}{4}} - \frac{r}{2}$$

$$\therefore y = \sqrt[3]{\sqrt{\frac{q^3}{27} + \frac{r^2}{4}} - \frac{r}{2}}.$$

What two numbers are thoſe whoſe product is 35, and difference of their third powers 218?

Let the leaſt number $= x$:

$\therefore$ the greateſt number $= \frac{35}{x}$

$$\therefore \left(\frac{35}{x}\right)^3 - x^3 = 218$$

$$\therefore \frac{42875}{x^3} - x^3 = 218$$

$$\therefore 42875 - x^6 = 218x^3$$

$$\therefore x^6 + 218x^3 = 42875$$

Let $z = x^3$

$$\therefore z^2 = x^6$$

$$\therefore z^2 + 218z = 42875$$

$$\therefore z^2 + 218z + \overline{109}^2 = 42875 + \overline{109}^2$$

$$\therefore \sqrt{z^2 + 218z + \overline{109}^2} = \sqrt{54756}$$

$$\therefore z + 109 = 234$$

$$\therefore z = 234 - 109 = 125$$

But $z = x^3$

$$\therefore x^3 = 125$$

$$\therefore x = \sqrt[3]{125} = 5.$$

What number is that, from whoſe third power if you take 19, and multiply the remainder by its third power, the product ſhall be 216 ?

Let x = the number:

$$\therefore \overline{x^3 - 19} \times x^3 = 216$$

$$\therefore x^6 - 19x^3 = 216.$$

Let $z = x^3$

$$\therefore z^2 - 19z = 216$$

$$\therefore z^2 - 19z + \frac{19^2}{4} = 216 + \frac{19^2}{4}$$

$$\therefore \sqrt{z^2 - 19z + \frac{19^2}{4}} = \sqrt{\frac{1225}{4}}$$

$$\therefore z - \frac{19}{2} = \frac{35}{2}$$

$$\therefore z = \frac{35}{2} + \frac{19}{2} = \frac{54}{2} = 27$$

$$\therefore x^3 = 27$$

$$\therefore x = 3.$$

A queſtion

A queſtion may involve two or more unknown ſimple numbers, from which an equation of the ſecond order will ariſe, and the ſolution depends, as in ſimple equations, upon finding by one equation the value of one unknown number in terms of the other unknown numbers, and ſubſtituting the value thus found for that unknown number in another equation. In finding theſe values, the learner may exerciſe his ingenuity, as there are various ways of coming to the ſame concluſion.

Let $x - \frac{x-y}{2} = 4$
and $y - \frac{x+3y}{x+2} = 1$
} What are the numbers x and y?

From the firſt equation, $2x - x + y = 8$:

$$\therefore x = 8 - y.$$

From the ſecond equation, $xy + 2y - x - 3y = x + 2$:

$$\therefore xy - 2x - y = 2.$$

Subſtitute in this laſt equation $8 - y$ for x:

$$\therefore \overline{8-y} \times y - 2 \times \overline{8-y} - y = 2$$
$$\therefore 8y - y^2 - 16 + 2y - y = 2$$
$$\therefore 9y - y^2 = 16 + 2 = 18$$
$$\left.\frac{9}{2}\right]^2 - 9y + y^2 = \left.\frac{9}{2}\right]^2 - 18$$
$$\therefore \sqrt{\left.\frac{9}{2}\right]^2 - 9y + y^2} = \sqrt{\frac{81-72}{4}} = \sqrt{\frac{9}{4}}$$

$\therefore$

$$\therefore \frac{9}{2} - y = \frac{3}{2}$$

$$\therefore y = \frac{9}{2} - \frac{3}{2} = \frac{6}{2} = 3$$

$$\therefore y = 3, x = 8 - 3 = 5.$$

Let $x^2 + y^2 = 65$ and $xy = 28$ } What numbers are x and y?

From the ſecond equation, $2xy = 56$.

Add this latter equation to the firſt.

$$\therefore x^2 + 2xy + y^2 = 121$$

$$\therefore \sqrt{x^2 + 2xy + y^2} = \sqrt{121}$$

$$\therefore x + y = 11.$$

Take away $2xy = 56$ from the firſt equation.

$$\therefore x^2 - 2xy + y^2 = 9$$

$$\therefore \sqrt{x^2 - 2xy + y^2} = \sqrt{9}$$

$$\therefore x - y = 3$$

or $y - x = 3$.

By the firſt $x = 3 + y$

by the ſecond $x = y - 3$

but

but $x + y = 11$

$\therefore 3 + y + y = 11$

$\therefore 2y = 11 - 3 = 8$

$\therefore y = 4$

$\therefore 3 + y = 3 + 4 = 7 = x.$

Let $x = y - 3$:

then $y - 3 + y = 11.$

$\therefore 2y = 11 + 3 = 14$

$\therefore y = 7$

$\therefore y - 3 = x = 7 - 3 = 4.$

If $x = 4$, then $y = 7$

If $y = 4$, then $x = 7.$

Let $x^2 + xy = 12$ and $xy - 2y^2 = 1$ } What are the numbers x and y?

Suppoſe vy to equal x.

Then $v^2y^2 + vy^2 = 12$

$\therefore y^2 = \frac{12}{v^2 + v}$

and $vy^2 - 2y^2 = 1$

$\therefore y^2 = \frac{1}{v - 2}$

$\therefore \frac{12}{v^2 + v} = \frac{1}{v - 2}$

$\therefore 12v - 24 = v^2 + v$

$\therefore 12v - v^2 - v = 24$

$$\therefore 11v - v^2 = 24$$

$$\therefore \left(\frac{11}{2}\right)^2 - 11v + v^2 = \left(\frac{11}{2}\right)^2 - 24$$

$$\therefore \sqrt{\left(\frac{11}{2}\right)^2 - 11v + v^2} = \sqrt{\frac{121}{4} - 24} = \sqrt{\frac{25}{4}}$$

$$\therefore \frac{11}{2} - v = \frac{5}{2}$$

$$\therefore v = \frac{11}{2} - \frac{5}{2} = \frac{6}{2} = 3$$

$$\text{or } v - \frac{11}{2} = \frac{5}{2}$$

$$\therefore v = \frac{11}{2} + \frac{5}{2} = \frac{16}{2} = 8$$

$$y^2 = \frac{1}{v-2} = \frac{1}{1} \text{ or } \frac{1}{6}$$

$$\therefore y = 1, \text{ or } \frac{1}{\sqrt{6}}$$

$$\therefore x = vy = 3 \text{ or } \frac{8}{\sqrt{6}}$$

When $y = 1$, $x = 3$.

When $y = \frac{1}{\sqrt{6}}$, $x = \frac{8}{\sqrt{6}}$.

Let $\frac{x^2}{y} + \frac{y^2}{x} = 18$ and $x + y = 12$ } What are the numbers x and y?

Make $x = z + v$

and $y = z - v$.

$$\therefore x + y = 2z = 12$$

$$\therefore z = \frac{12}{2} = 6$$

$$\therefore x = 6 + v$$

and $y = 6 - v$.

$$\text{But } \frac{x^2}{y} + \frac{y^2}{x} = 18$$

$$\therefore x^3 + y^3 = 18xy$$

$$\text{But } x^3 = \overbrace{6 + v}^{3} = 216 + 108v + 18v^2 + v^3$$

$$\text{and } y^3 = \overbrace{6 - v}^{3} = 216 - 108v + 18v^2 - v^3$$

$$\therefore x^3 + y^3 = 432 + 36v^2$$

$$xy = \overbrace{6 + v} \times \overbrace{6 - v} = 36 - v^2$$

$$\therefore 18xy = 648 - 18v^2$$

$$\text{But } x^3 + y^3 = 18xy$$

$$\therefore 432 + 36v^2 = 648 - 18v^2$$

$$\therefore 36v^2 + 18v^2 = 648 - 432$$

$$\therefore 54v^2 = 216$$

$$\therefore v^2 = \frac{216}{54} = 4$$

$$\therefore v = 2$$

$$x = 6 + v = 6 + 2 = 8$$

$$y = 6 - v = 6 - 2 = 4$$

What two numbers are they whose product is equal to a, and the sum of their squares to b?

Let x and y be the numbers.

Then $xy = a \quad \therefore x = \frac{a}{y}$

and $x^2 + y^2 = b \quad \therefore x^2 = b - y^2$

$$\therefore \frac{a^2}{y^2} = b - y^2$$

$$\therefore a^2 = by^2 - y^4.$$

Let $z = y^2$.

$$\therefore a^2 = bz - z^2$$

$$\therefore \frac{b^2}{4} - a^2 = \frac{b^2}{4} - bz + z^2$$

$$\therefore \sqrt{\frac{b^2}{4} - a^2} = \frac{b}{2} - z, \text{ or } z - \frac{b}{2}$$

$$\therefore z = \frac{b}{2} - \sqrt{\frac{b^2}{4} - a^2} \text{ or } \frac{b}{2} + \sqrt{\frac{b^2}{4} - a^2}$$

$$\therefore y^2 = \frac{b}{2} \pm^{*} \sqrt{\frac{b^2}{4} - a^2}$$

$$\therefore y = \sqrt{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}}$$

$$\therefore x = \frac{a}{y} = \frac{a}{\sqrt{\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}}}$$

* Inſtead of writing down the two values of y^2 at length, the marks $\pm$ are inſerted between the two terms to which y^2 is equal, and the ſame is done for the two correſponding values of x: and caution only is neceſſary, that if the mark of addition is taken in the term for y, the ſame may be done in the term for x. Or, in other words, both upper marks are to be uſed together, and both the lower marks; but if the upper mark is uſed for the value of y, the lower mark cannot be uſed for the value of x.

ON

ON EQUATIONS OF THE THIRD ORDER.

WHEN the higheſt power of the unknown number in an equation is the third, the equation is of the third order. And of this order there are the following forms:

$$x^3 = c$$

$$x^3 + ax^2 = c$$

$$x^3 - ax^2 = c$$

$$x^3 + bx = c$$

$$x^3 - bx = c$$

$$x^3 + ax^2 + bx = c$$

$$x^3 - ax^2 + bx = c$$

$$x^3 + ax^2 - bx = c$$

$$x^3 - ax^2 - bx = c$$

$$ax^2 - x^3 = c$$

$$ax^2 + bx - x^3 = c$$

$$ax^2 - bx - x^3 = c$$

$$bx - ax^2 - x^3 = c$$

$$bx - x^3 = c.$$

The ſolution of equations of this order is difficult, and a general mode, which ſhall anſwer as in equations of the ſecond order, has not been hitherto diſcovered. For, by adding

adding numbers to both ſides, which ſhall on one ſide make a compound term, correſponding to the third power of a three-fold term, the unknown term will be involved on the other ſide, and the ſolution of the equation will not be promoted. Thus, in the form $x^3 + ax^2 = c$, by adding $bx + e^3$ to both ſides a new equation is formed $x^3 + ax^2 + bx + e^3 = c + bx + e^3$. Therefore $\sqrt[3]{x^3 + ax^2 + bx + e^3} = \sqrt[3]{c + bx + e^3}$. The third root of $x^3 + ax^2 + bx + e^3$ may be $x + e$; but though, from having the co-part of x^2 known, we know the amount of b and e^3, which will anſwer in this caſe; and conſequently know the term e in the two-fold term $x + e$: yet, on the other ſide of the equation, there remains a third root to be found, namely, that of $c + bx + e^3$, which cannot be done, becauſe there is one unknown term in it, namely, bx. The ſame is true of all the other forms except the firſt, and hence various methods have been propoſed to reſolve theſe equations, of which the beſt is by the mode of dividers, a mode applicable to equations of all orders. It will be eaſily underſtood by inveſtigating it in equations of the ſecond order. Thus, let

$$x^2 + ax = b$$

$$\therefore x \times \overline{x + a} = b$$

$$\therefore x + a = \frac{b}{x}$$

$$\text{and } x = \frac{b}{x + a}.$$

Hence b is diviſible by both x and $x + a$.

Let $b = cd$ or ef or gh or ik, &c.; for a number b may be the product of the multiplication of various ſets of num-

bers. Then $x = \frac{cd}{x+a}$; and if $x + a = c$, x will be equal to d. Or, if x is equal to c, then $x + a$ is equal to d. Among the combinations of numbers into which b may be divided, two will be found equal to x and $x + a$; and thus an equation may be folved by fubſtituting one of the dividers of the known ſide for the unknown term on the other, till the right value of the unknown term is found.

Let $x^2 + 20x = 96$:

48	32	24	16	12	} Dividers of 96.
2	3	4	6	8	}

If any term in the lower ſeries $= x$, the term above it in the upper ſeries $= x + 20$.

Let 2 be ſuppoſed equal to x, then $x + 20 = 22$; but $x + 20$ in this caſe ſhould be 48, and conſequently 2 cannot be equal to x. Let 4 now be tried, then $x + a = 4 + 20 = 24 =$ the number above 4, which is conſequently the true value of x. For $24 \times 4 = \overline{x+a} \times x = 96$.

In the ſame manner with the other forms of the ſecond order, the equation may be ſolved by finding among the dividers of the known ſide that which anſwers to the conditions of the queſtion. In equations of the higher orders the ſame method may be applied with equal certainty. Let $x^3 + ax^2 = c$. Then $x^2 \times \overline{x + a} = c$. Among the dividers of c will be found two correſponding to x^2 and $\overline{x+a}$, from which x is ſometimes eaſily found.

Let

Let $x^3 + x^2 = 36$:

$\therefore x^2 \times \overline{x+1} = 36$

$$\left.\begin{matrix} 18 & 12 & 9 & 6 \\ 2 & 3 & 4 & 6 \end{matrix}\right\} \text{Dividers of } 36.$$

In the upper ſeries are numbers to be ſubſtituted for x^2; in the lower, numbers for $x + 1$. Now the third root of 36 lies between 3 and 4: therefore x^2 muſt be leſs than 16, and the dividers 18 and 2 are excluded. Again, if $x + 1 = 6$, $x = 5$, which cannot be in this equation, and conſequently 6 and 6 are excluded. There remain therefore to be conſidered 9×4 and 12×3. But if the latter product is taken $x + 1 = 3$, and $x = 2$; but 12 is greater than the ſecond power of 2, and conſequently 12×3 muſt be excluded. There remains then only 9×4, and if $9 = x^2$, $x = 3$, and $x + 1 = 3 + 1 = 4$, and conſequently $x = 3$.

Let $x^3 + 3x^2 - 16x = 48$:

$\therefore x \times \overline{x^2 + 3x - 16} = 48.$

$$\left.\begin{matrix} 24 & 16 & 12 & 8 \\ 2 & 3 & 4 & 6 \end{matrix}\right\} \text{Dividers of } 48.$$

The product 8×6 is excluded; becauſe, if $x = 6$, $x^2 + 3x - 16$ is greater than 8. On examining the product 12×4, it appears that if $x = 4$, $x^2 + 3x - 16 = 12 =$ number above 4 in the upper ſeries, and conſequently $x = 4$.

Let $4x^2 - 30x + x^3 = 75$:

$\therefore x \times \overline{4x - 30 + x^2} = 75.$

$$\left.\begin{matrix} 1 & 3 & 5 \\ 75 & 25 & 15 \end{matrix}\right\} \text{Dividers of } 75.$$

The products 1×75 and 3×25 are rejected, becauſe if $x = 1$ or 3, $4x + x^2$ will be leſs than 30. Let $5 \times 15 = x \times \overline{4x - 30 + x^2}$, and $x = 5$; and $4x - 30 + x^2 = 15$.

Let $x^3 - 4x^2 + 5x - 2 = 0$:

$\therefore x^3 - 4x^2 + 5x = 2$

$\therefore x \times \overline{x^2 - 4x + 5} = 2.$

$$\left.\begin{matrix} 1 \\ 2 \end{matrix}\right\} \text{Dividers of } 2.$$

Therefore 2×1 is the product in which x is to be found. But if $x = 1$, then $x^2 - 4x + 5 = 2$; and if $x = 2$, then $x^2 - 4x + 5 = 1$. Therefore this equation has two ſolutions, x being equal to either 1 or 2.

Let $x^3 + 6x - 20 = 0$:

$\therefore x^3 + 6x = 20$

$\therefore x \times \overline{x^2 + 6} = 20.$

$$\left.\begin{matrix} 20 & 10 & 5 \\ 1 & 2 & 4 \end{matrix}\right\} \text{Dividers of } 20.$$

10×2 is the product $= x \times \overline{x^2 + 6}$; for if $x = 2$, $x^2 + 6 = 10$. $\therefore x = 2$.

Let

Let $x^3 - 12x^2 + 41x - 42 = 0$:

$\therefore x^3 - 12x^2 + 41x = 42$

$\therefore x \times \overline{x^2 - 12x + 41} = 42.$

42	21	14	7	} Dividers of 42.
1	2	3	6	

If $x = 2$, $x^2 - 12x + 41 = 21$

$x = 3$, $x^2 - 12x + 41 = 14$

$x = 7$, $x^2 - 12x + 41 = 6$

$\therefore x = 2$, 3, or 7.

Let $x^3 + 15x^2 + 84x = 100$:

$\therefore x \times \overline{x^2 + 15x + 84} = 100.$

100	50	25	20	10	} Dividers of 100.
1	2	4	5	10	

All the products except that of 100×1 muſt be rejected at firſt ſight; for 84×2 is greater than 100; and if $x = 1$, $x^2 + 15x + 84 = 100$. $\therefore x = 1$.

If x in the propoſed equation is a whole number, it will be found in the whole dividers of the known ſide of the equation; but x may contain a fraction, and in that caſe it cannot be found, unleſs the fractional dividers as well as the others are written down. This would be very tedious; and, to ſhorten the labour, the neareſt limits in whole numbers within which x is confined, may be found; and thence an approximation may be made, by transforming the propoſed equation into a new one, in which the un-

known number ſhall be the difference between the limit and x.

$$\text{Let } x^3 - ax^2 + bx = c.$$

Among the dividers of c let d and e, differing by unity, be the one greater and the other leſs than x. Suppoſe therefore x to be equal to $d - y$ or $e + y$: by ſubſtituting either of theſe values for x in the propoſed equation, a new one ariſes, from which y may be diſcovered.

$$\left.\begin{array}{l} d^3 - 3dy + 3dy^2 - y^3 \\ - ad^2 + 2ady - ay^2 \\ + bd - by. \end{array}\right\} = c.$$

$$\left.\begin{array}{l} \text{or } e^3 + 3ey + 3ey^2 + y^3 \\ - ae^2 - 2aey - ay^2 \\ + be + by \end{array}\right\} = c.$$

$$\text{Let } x + x^3 = \frac{5}{8}$$

In this equation x is evidently leſs than one.

$$\text{Let } x = 1 - y.$$

$$\therefore 1 - y + 1 - 3y + 3y^2 - y^3 = \frac{5}{8}$$

$$\therefore 2 - \frac{5}{8} = y^3 + 4y - 3y^2$$

$$\therefore y \times \overline{y^2 + 4 - 3y} = \frac{11}{8}$$

$\frac{1}{2}$

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ Dividers of $\frac{11}{8}$

$\frac{11}{4}$ $\frac{33}{8}$ $\frac{11}{2}$ Co-parts of the dividers.

Let $y = \frac{1}{2}$, then $y^2 + 4 - 3y = \frac{1}{4} + 4 - \frac{3}{2} = \frac{11}{4} =$ co-part of $\frac{1}{2}$. Therefore $y = \frac{1}{2}$, and $x = 1 - \frac{1}{2} = \frac{1}{2}$.

Let $x + x^3 = \frac{2}{3}$.

$$\therefore x \times \overline{1 + x^3} = \frac{2}{3}$$

$\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ ..... Dividers.

$\frac{4}{3}$ 2 $\frac{8}{3}$ $\frac{10}{3}$ 4 ..... Co-parts.

Let $x = \frac{1}{2}$, then $1 + x^2 = 1 + \frac{1}{4} = \frac{5}{4}$ = a number leſs than $\frac{4}{3}$: Therefore x is not equal to, but greater than, one-half; and conſequently all the numbers in the upper ſeries may be rejected. And ſince x is leſs than one, and greater than $\frac{1}{2}$, ſome number may be found between theſe limits, to which it is equal.

Let this number be $\frac{1}{2} + y$.

$$\therefore \frac{1}{2} + y + \overline{\frac{1}{8} + \frac{3}{4}y + \frac{3}{2}y^2 + y^3} = \frac{2}{3}$$

$$\therefore y \times \frac{7}{4} + \frac{3}{2}y + y^2 = \frac{1}{24}.$$

Hence y is evidently leſs than $\frac{1}{24}$; and ſince $\frac{7}{4}$ is in the co-part of y, that co-part muſt be greater than $\frac{7}{4}$. If the co-part were $\frac{7}{4}$, then $y \times \frac{7}{4}$ would be equal to $\frac{1}{24}$, or y would be equal to $\frac{4}{24 \cdot 7} = \frac{1}{42}$. Therefore y muſt be leſs than $\frac{1}{42}$. Conſequently all the numbers preceding $\frac{1}{42}$, in the upper ſeries, may be rejected.

$\frac{1}{42}$ $\frac{1}{43}$ $\frac{1}{44}$ $\frac{1}{45}$ $\frac{1}{46}$ ... Dividers.

$\frac{42}{24}$ $\frac{43}{24}$ $\frac{44}{24}$ $\frac{45}{24}$ $\frac{46}{24}$ .... Co-parts.

Let $y = \frac{1}{43}$, then $\frac{7}{4} + \frac{3}{2}y + y^2 = \frac{43}{24}$

$$\therefore y \times \overline{\tfrac{3}{2} + y} = \frac{43}{24} - \frac{7}{4} = \frac{1}{24}.$$

$\therefore \frac{3}{2} + y = \frac{43}{24}$, which is impoſſible; for $\frac{7}{4} + \frac{3}{2}y + y^2$ is only equal to $\frac{43}{24}$. Therefore y is leſs than $\frac{1}{42}$, and greater than $\frac{1}{43}$.

$$\therefore x < \frac{1}{2} + \frac{1}{43} \text{ or } \frac{45}{86}.$$

Make $x = \frac{45}{86} + z$.

Then

Then $\frac{45}{86} + z + \overline{\frac{45}{86}}^3 + 3.\overline{\frac{45}{86}}^2 z + 3.\frac{45}{86}z + z^3 = \frac{2}{3}$

$$\therefore z \times 1 + \overline{3.\frac{45}{86}}^2 + \frac{3.45}{86}z + z^2 = \frac{2}{3} - \frac{45}{86} - \overline{\frac{45}{86}}^3$$

and from this equation, by proceeding as before, z may be nearly found, and from thence a nearer number to x.

Let $28x - 24x^2 + 8x^3 = 15$.

Since $x \times \overline{28 - 24x + 8x^2} = 15$; if $x = 2$,

$28 - 24x + 8x^2 = \frac{15}{2}$. But $28 - 24x + 8x^2$

in this caſe is equal to 12; therefore 2 is too great a number: and if 1 is ſubſtituted for x, the unknown ſide of the equation is too ſmall. Therefore x muſt lie between 1 and 2.

Let $x = 1 + y$.

$$\left.\begin{array}{r} 28+28y \\ -24-48y-24y^2 \\ +\ 8+24y+24y^2+8y^3 \end{array}\right\} = 15$$

$$\therefore 12 + 4y + 8y^3 = 15$$

$$\therefore y \times \overline{4 + 8y^2} = 3.$$

1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	Dividers.
3	6	9	12	15	Co-parts.

Let

Let $y = \frac{1}{2}$: then $4 + 8y^2 = 4 + \frac{8}{4} = 4 + 2 = 6$ the number under $\frac{1}{2}$ in the lower ſeries.

$$\therefore y = \tfrac{1}{2}, \text{ and } x = 1\tfrac{1}{2}.$$

Let $x^3 + 2x^2 + 3x = 24{,}381$.

x is evidently greater than one, and leſs than three. If $x = 2$, its co-part $x^2 + 2x + 3 = 11$, and the product is only 22 : conſequently x is greater than 2.

Let $x = 2 + y$.

$$\left.\begin{array}{l} \therefore 8 + 12y + 6y^2 + y^3 \\ +\ 8 +\ \ 8y + 2y^2 \\ +\ 6 +\ \ 3y \end{array}\right\} = 24{,}381$$

$$\therefore 22 + 23y + 8y^2 + y^3 = 24{,}381$$

$$\therefore y \times \overline{23 + 8y + y^2} = 2{,}381$$

$$\begin{array}{llll} ,1 & ,2 & ,3 \ldots . & \text{Dividers.} \\ 23{,}81 & 11{,}905 & 7{,}936\tfrac{6}{3} \ldots . & \text{Co-parts.} \end{array}$$

If $y = ,1$, $y \times \overline{8 + y} = ,81 = ,1 \times \overline{84 + ,1}$

$\therefore$ the co-part of y = co-part of ,1, and $y = ,1$

$$\therefore x = 2{,}1.$$

Let $41x - 3x^2 - 5x^3 = 12$.

$$\therefore x \times \overline{41 - 3x - 5x^2} = 12$$

$$\left.\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 12 & 6 & 4 & 3 \end{array}\right\} \text{Dividers.}$$

x is

x is evidently greater than one. If $x = 2$, its co-part is greater than 6; and if $x = 3$, its co-part cannot be expreſſed in numbers. Therefore x is greater than two, and leſs than three.

Let $x = 2 + y$.

$$\left.\begin{array}{l} \therefore 82 + 41y \\ -12 - 12y - 3y^2 \\ -40 - 60y - 30y^2 - 5y^3 \end{array}\right\} = 12$$

$$\therefore 30 - 31y - 33y^2 - 5y^3 = 12$$

$$y \times \overline{31 + 33y + 5y^2} = 18$$

,1 ,2 ,3 ,4 ,5 ,6 Dividers.

180 90 60 45 36 30 Co-parts.

y is leſs than ,6, becauſe the number under 6 is leſs than one term in the co-part of y; it is alſo leſs than ,5, for $31 + 33y$ is greater than 36. Let $y = ,4$, and then $31 + 33y + 5y^2 = 31 + 13,2 + ,8 = 45$.

$$\therefore y = ,4, \text{ and } x = 2,4.$$

It is uſeful, frequently to change the propoſed equation into another which ſhall have a ſmaller number of terms in it, and conſequently the dividers of the known ſide may be with greater eaſe confined within certain limits. Thus, if $x^3 + ax^2 + bx = c$. In examining the dividers of c, it is neceſſary to compare their co-parts with $x^2 + ax + b$: but if the equation could be changed into another of this form $y^3 + by = d$, in which the difference between x and y is a given quantity; y might be found with greater eaſe among the dividers of d, ſince its co-part $y^2 + b$ is ſimpler than the co-part of x, in the given equation.

Let

Let $x^3 + ax^2 + bx = c$, and make $x = y - d$.

$$\text{Then } \left.\begin{array}{r} y^3 - 3dy^2 + 3d^2y - d^3 \\ + ay^2 - 2ady + ad^2 \\ + by - bd \end{array}\right\} = c$$

In the new equation, if d is such a number that $ay^2 = 3dy^2$, there will not be an unknown number in the second power; for $ay^2 - 3dy^2$ in that case will be equal to nothing. But if $ay^2 = 3dy^2$, $d = \frac{a}{3}$. Therefore if $x = y - \frac{a}{3}$, a new equation may be formed without the second power of the unknown term. $\frac{a}{3}$ is the third part of the known co-part of the second power of the unknown number in the propoſed equation; and conſequently, by dividing the known co-part of the term having the ſecond power of the unknown number by three, a number is found, by means of which the new equation is to be formed.

Let $x^3 + 3x^2 + 2x = 24$.

Make $x = y - \frac{3}{3} = y - 1$.

$$\text{Then } \left.\begin{array}{r} y^3 - 3y^2 + 3y - 1 \\ + 3y^2 - 6y + 3 \\ + 2y - 2 \end{array}\right\} = 24$$

$$\therefore y^3 - y = 24$$

$$\therefore y \times \overline{y^2 - 1} = 24.$$

$$\left.\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 24 & 12 & 8 & 6 \end{array}\right\} \text{Dividers.}$$

Let

Let $y = 3$; then $y^2 - 1 = 8$, and $y^2 = 9$.

$$\therefore y = 3$$

$$\therefore x = 3 - 1 = 2.$$

If the propoſed equation had not been changed, among the dividers of 24, one whoſe co-part is equal to $x^2 + 3x + 2$, muſt have been ſought for.

Upon the ſame principle an equation may be changed into another, in which the term containing the ſimple unknown number ſhall be wanting. Thus, in the new equation made from $x^3 + ax^2 + bx = c$, by changing x into $y - d$, the co-part of y is $3d^2 - 2ad + b$; and conſequently, if $3d^2 + b = 2ad$, the term having y for one part will be wanting. This can ſeldom be of any uſe; as, to find d, an equation of the ſecond order muſt be ſolved.

Thus many equations of the third order may be ſolved with great eaſe: but ſometimes the number of dividers of the known term is very conſiderable, and alſo the unknown number may have a fractional value. In theſe caſes it is uſeful to transform the equation into two others of the ſame value, in which the unknown numbers ſhall be greater or leſs than the unknown number in the given equation by a given quantity; and from a compariſon of the dividers in the three equations, the unknown number in the given equation may with greater eaſe be diſcovered.

Let $x^3 - ax^2 + bx = c$.

Make $x = y + 1$.

$$\left.\begin{array}{rl} \therefore x^3 = & y^3 + 3y^2 + 3y + 1 \\ -ax^2 = & \quad - ay^2 - 2ay - a \\ +bx = & \quad\quad + by + b \end{array}\right\} = c.$$

$\therefore$

$$\left.\begin{array}{r} \therefore y^3 + 3y^2 + 3y \\ - ay^2 - 2ay \\ + by \end{array}\right\} = c - b + a - 1.$$

Hence a value of y is to be found among the dividers of $c - b + a - 1$, which is leſs than that of x by 1.

Make now $x = z - 1$.

$$\left.\begin{array}{l} \therefore x^3 = \\ -ax^2 = \\ +bx = \end{array}\right\} \left.\begin{array}{r} z^3 - 3z^2 + 3z - 1 \\ - az^2 + 2az - a \\ + bz - b \end{array}\right\} = c.$$

$$\left.\begin{array}{r} \therefore z^3 - 3z^2 + 3z \\ - az^2 + 2az \\ + bz \end{array}\right\} = c + b + a + 1.$$

A value of z is to be found among the dividers of $c + b + a + 1$, which is greater than that of x by 1. Write down therefore the dividers of $c - b + a - 1$, c, $c + b + a + 1$, in three ſeries, and a value of x will be found in that divider of c, which has, in the ſeries above and below, a number leſs and greater than itſelf by unity.

Let $x^3 - 3x^2 + 2x = 24$.

$24 - 2 + 3 - 1 = 24$	1	2	3	4	6	8	12	24
$24 = 24$	1	2	3	4	6	8	12	24
$24 + 2 + 3 + 1 = 30$	1	2	3	5	6	10	15	30.

2 and

2 and 4 are two dividers of 24, between two divi... of the upper and lower ſeries 1 and 3, and 3 and 5 greater than the one in the upper, and leſs than that in the lower ſeries, by unity. Therefore, if the unknown number is among theſe dividers, it muſt be either 2 or 4. If 2 is taken for the number, its co-part muſt be $12 = x^2 - 3x + 2$. But $x^2 + 2$ is only equal to 6. $\therefore$ 2 does not give the ſolution, but 4 does; for its co-part is 6, and $6 = x^2 - 3x + 2 = 16 - 12 + 2$.

In the two new equations having the unknown terms y and z, it appears, that on carrying over the known quantities to one ſide, a compound term is to be taken away from c, the known term in the given equation, and that this compound term conſiſts of the known co-parts of the unknown terms. The marks however before theſe co-parts do not remain the ſame as in the given equation. When in the new equation the unknown number is leſs than that of the unknown number in the given equation by unity, the known terms ariſing on one ſide of the equation from ſubſtituting $y + 1$ for x, are the known co-parts of the unknown terms with the ſame marks. Thus, $1 - a + b$ in the firſt transformed equation is made of the known co-parts of $x^3 - ax^2 + b$, and the marks in both compound terms are the ſame. Conſequently, in taking away this compound term from c, the marks before the known co-parts muſt be changed, and the known term, which has a divider leſs than x by unity, is $c - 1 + a - b$.

In the ſecond equation, in which the unknown number is greater than that in the given equation by unity, the known numbers ariſing from making $x = z - 1$ are as before

before the known co-parts of the unknown terms in the given equation; but they are all to be taken away on one ſide of the equation; and therefore, on being tranſpoſed to the other ſide of the equation, they are to be added to c, and conſequently in the dividers of $c + 1 + a + b$, one is greater than a divider of c by 1.

For the other forms of equations of the third order, three ſeries of dividers are to be found in the ſame manner, by making two new equations; and the compound term to be taken from c will, in all caſes, be made of the co-parts of the unknown terms; but the marks of this new compound term will depend on the form of the propoſed equation. Firſt, the compound term to be taken from c, to give dividers in which one is leſs by unity than the unknown number in the propoſed equation, will have its marks the ſame as thoſe of the terms in the propoſed equation. Secondly, For the compound term to be taken from c, to give dividers in which one is greater than the unknown number in the propoſed equation by unity, the marks of the co-parts will differ from thoſe in the propoſed equation. For the co-part of the unknown terms, whoſe index is either three or one, will have the oppoſite mark to that which it had in the propoſed equation.

Thus, in the form $x^3 - ax^2 + bx = c$, if $x = z - 1$, the new known term reſulting will be $-1 - a - b$, which being tranſpoſed, will make the other ſide of the equation $c + 1 + a + b$. This proceeds from the nature of the marks in a two-fold term raiſed to the third power; for its laſt term will always have the ſame mark as the ſecond term in the two-fold term from which it

was taken. The laſt term of $\overline{z-1}^3$ is -1; of $\overline{z+1}^3$ is $+1$. The mark of the term, on the contrary, of a two-fold term, raiſed to the ſecond power, never varies: for the laſt term is always to be added to the other terms, whether in the ſimple two-fold term the ſecond term was to be added or taken away. The laſt term in $z-1$, raiſed to the ſecond power, is $+1$, as is alſo the laſt term in $z+1^2$.

Hence, as in tranſpoſing terms they change their marks, the compound term, whoſe dividers are to be found, may be written down with great eaſe. The term, one of whoſe dividers is leſs by unity than x, will be c followed by the co-parts of the unknown term in the given equation, with their marks changed. The term one of whoſe dividers is greater than x by unity will be c, followed by the co-parts of the unknown term; that of x^2 only having its mark changed.

Thus, let the equation $x^3 + ax^2 = c$ be propoſed for reſolution by dividers; then

$$\left.\begin{matrix} c-a-1 \\ c \\ c-a+1 \end{matrix}\right\} \text{contains a divider} \left\{\begin{matrix} \text{leſs than} \\ \text{equal to} \\ \text{greater than} \end{matrix}\right\} x \left\{\begin{matrix} \text{by unity.} \\ \\ \text{by unity.} \end{matrix}\right.$$

For the dividers of the equation $x^3 + ax^2 - bx = c$, the terms are $c + b - a - 1$, c, $c - b - a + 1$.

For the dividers of the equation $x^3 - ax^2 - bx = c$, the terms including dividers are $c + b + a - 1$, c, $c - b + a + 1$.

For the dividers of the equation $ax^2 - bx + x^3 = c$, the terms for dividers are $c - 1 + b - a$, c, $c + 1 - b - a$.

By obſerving the rule, the dividers for the other forms may be written down without trouble.

QUESTIONS

PRODUCING EQUATIONS OF THE THIRD ORDER.

IN the ſolution of the following queſtions, all the dividers of the middle term c are not ſet down; as from a ſimple view of the equation, it will appear that ſome cannot be applied; and by taking the dividers of the middle term firſt, it will be ſeen what may be taken from the dividers of the other terms.

Some officers and ſoldiers were travelling together. The officers had each twice as many guineas as there were ſoldiers in company, and for each officer there were as many ſoldiers as there were officers in the company; and their money amounted to 3456 guineas. What was the number of officers and ſoldiers?

Let x = the number of officers,

then x^2 = the number of ſoldiers:

$\therefore 2x^2$ = number of guineas which each man had,

$\therefore 2x^2 \times x = 2x^3$ = number of guineas,

$\therefore 2x^3 = 3456.$

$\therefore$

$$\therefore x^3 = 1728$$

$$\therefore \sqrt[3]{x^3} = \sqrt[3]{1728}$$

$$\therefore x = 12.$$

There were 12 officers and 144 ſoldiers in company.

By dividing ſixteen pounds between two perſons, the third power of one's ſhare exceeded the third power of the other's by 386. What were their ſhares?

Let one ſhare $= x$

then the other ſhare $= 16 - x$

$$\therefore x^3 - \overline{16 - x}|^3 = 386$$

$$\therefore x^3 - 4096 + 768x - 48x^2 + x^3 = 386$$

$$\therefore 2x^3 + 768x - 48x^2 = 386 + 4096 = 4482$$

$$\therefore x^3 + 384x - 24x^2 = 2241$$

$2241 + 24 - 384 - 1 = 1880$
2241
$2241 + 24 + 384 + 1 = 2650$

} 2, 4, 5, 8, 3, 9 ..... 249,747 5, 10 ...

9 is the divider, having a number above and below differing from it by unity; and if $x = 9$, $x^2 + 384 - 24x = 249$.

$$\therefore x = 9, \text{ and } 16 - x = 7.$$

The two ſhares were 9 and 7 pounds.

The dividers between 9 and 249 were not written down, becauſe it was evident that the next divider to 9 muſt be greater than the value of x.

What two numbers are thoſe, the ſum of whoſe ſquares is 58, and their product, multiplied by the greater, 147 ?

Let x be one, and y = the leaſt number:

$$\therefore x^2 + y^2 = 58$$
$$\therefore x^2 = 58 - y^2$$
$$\text{and } xy \times x = 147$$
$$\therefore x^2 = \frac{147}{y}$$
$$\therefore 58 - y^2 = \frac{147}{y}$$
$$\therefore y \times \overline{58 - y^2} = 147$$

$147 + 1 - 58 = 90$ } 1, 2, 3, 5, 6....
147 } 1, 3, 7, 21, 49, 147
$147 - 1 + 58 = 204$ } 1, 2, 3, 4, 6....

y cannot be a whole number greater than 7; for if it were, y^2 could not be taken from 58.

3 lies between the two numbers 2 and 4, to be found in the upper and lower ſeries. And if $y = 3$, $58 - y^2 = 49$.

$$\therefore y = 3, \text{ and } x^2 = \frac{147}{3} = 49$$
$$\therefore x = 7.$$

A, B, and C had amongſt them 2000 ſhillings, of which B gave to A a number of ſhillings equal to the ſecond root of A's ſhare, and had 341 ſhillings left; but if he had given to C a number equal to the third root of C's ſhare, he

he would have had 362 ſhillings left. What was each man's ſhare at firſt?

A's ſhare $= x$, B's $= y$, C's $= z$.

$\therefore x + y + z = 2000$, $\therefore y = 2000 - x - z$

$y - x^{\frac{1}{2}} = 341$ $\therefore y = 341 + x^{\frac{1}{2}}$

$y - z^{\frac{1}{3}} = 362$ $\therefore y = 362 + z^{\frac{1}{3}}$

$\therefore 341 + x^{\frac{1}{2}} = 362 + z^{\frac{1}{3}}$

$\therefore x^{\frac{1}{2}} = 21 + z^{\frac{1}{3}}$

$\therefore x = 441 + 42z^{\frac{1}{3}} + z^{\frac{2}{3}}$.

But $362 + z^{\frac{1}{3}} = 2000 - x - z$

$\therefore 362 + z^{\frac{1}{3}} = 2000 - 441 - 42z^{\frac{1}{3}} - z^{\frac{2}{3}} - z$

$\therefore z^{\frac{1}{3}} + 42z^{\frac{1}{3}} + z^{\frac{2}{3}} + z = 1197$

$\therefore z + z^{\frac{2}{3}} + 43z^{\frac{1}{3}} = 1197$.

Let $z = v^3$.

$\therefore v^3 + v^2 + 43v = 1197$

and v muſt be leſs than $\sqrt[3]{1197}$ or 11.

$1197 - 1 - 1 - 43 = 1152$ } 1. 2. 3. 4. 6. 8. 9
1197 } 1. 3. 7. 9.
$1197 + 1 - 1 + 43 = 1240$ } 1. 2. 4. 5. 8. 10

$v = 9$ $\therefore v^3 = 729 = z$

$\therefore 729$ = C's ſhare

371 = B's ſhare

990 = A's ſhare.

A bought ſeven ſheep more than B for 15*l.* and the number of B's ſheep, raiſed to the ſecond power, is equal to the number of ſhillings that A paid for each ſheep. How many did each buy?

$$\left.\begin{array}{l}\text{Let } x =\\ \therefore\ x+7 =\end{array}\right\}\text{ſheep bought by}\begin{cases}\text{B}\\ \text{A}\end{cases}$$

$$\therefore \frac{300s.}{x+7} = \text{ſhillings paid by A for each ſheep}$$

$$\therefore x^2 = \frac{300}{x+7}$$

$$\therefore x^3 + 7x^2 = 300$$

$$\left.\begin{array}{l}300 - 7 - 1 = 292\\ \qquad\qquad 300\\ 300 - 7 + 1 = 294\end{array}\right\}\begin{array}{l}4,\\ 4, 5, 6 \cdots 75, 60, 50\\ 6\end{array}$$

5 being equal to x, $x^2 + 7x = 60$

$\therefore$ 5 = number of ſheep bought by B

$\therefore x + 7 = 12$ = number of ſheep bought by A.

How is 35 to be divided into two parts, ſo that the ſecond power of the ſmaller part, multiplied into the other, may produce 750?

Let x = the leaſt part.

$\therefore 35 - x$ = the other part

$\therefore \overline{35-x} \times x^2 = 750$

$$\left.\begin{array}{l}750 - 35 + 1 = 716\\ \qquad\qquad 750\\ 750 - 35 - 1 = 714\end{array}\right\}\begin{array}{l}1; 2; 4\\ 2; 3; 5; 6;\\ 3; 6\end{array}$$

If

If $x = 5$, then the co-part of $x^2 = \frac{750}{25} = 30$

but $35 - x = 35 - 5 = 30$

$\therefore x = 5$, and $35 - x = 30$

The two parts are 5 and 30.

What two numbers are as three to two, when the ſum of their third powers is 280?

Let x = one number.

$\therefore 3 : 2 :: x : \frac{2x}{3}$ the other number

$\therefore x^3 + \frac{8x^3}{27} = 280$

$\therefore \frac{35x^3}{27} = 280$

$\therefore \frac{x^3}{27} = 8$

$\therefore \sqrt[3]{\frac{x^3}{27}} = \sqrt[3]{8}$

$\therefore \frac{x}{3} = 2$

$\therefore x = 6$, and $\frac{2x}{3} = 4$

The two numbers are 6 and 4.

Two maſons, A and B, jointly perform a piece of work in twelve days; and if the ſum of the days in which they could each have ſeparately performed it were multiplied into the days in which A, the beſt workman, could have done it, the product would be 1000. In what time could each do the work?

Let $x =$
 $y =$ } days in which $\left\{\begin{matrix} A \\ B \end{matrix}\right\}$ could do it

$x : 1 :: 12 : \frac{12}{x}$
 $y : 1 :: 12 : \frac{12}{y}$ } work done by $\left\{\begin{matrix} A \\ B \end{matrix}\right.$

$$\therefore \frac{12}{x} + \frac{12}{y} = 1$$

$$\therefore 12y + 12x = xy$$

$$\therefore 12x = xy - 12y = y \times \overline{x - 12}$$

$$\therefore y = \frac{12x}{x - 12}$$

but $\overline{x + y} \times x = 1000$

$$\therefore \overline{x + \frac{12x}{x-12}} \times x = 1000$$

$$x^2 \times x = 1000x - 12000$$

$$\therefore x \times \overline{1000 - x^2} = 12000$$

$$\left.\begin{matrix} 12000 - 1000 + 1 = 11001 \\ 12000 \end{matrix}\right\} 19$$

$$\left.\begin{matrix} 12000 + 1000 - 1 = 12999 \end{matrix}\right\} 21 \quad 12.16.20.....750.600$$

x evidently cannot be equal to, or less than, 12; and 20 has a number in each series, differing from it by unity, and its co-part $600 = 1000 - x^2$.

$$\therefore x = 20, \text{ and } y = \frac{12.20}{20 - 12} = \frac{240}{8} = 30.$$

EQUATIONS

EQUATIONS OF A HIGHER ORDER THAN THE THIRD.

EQUATIONS of a higher order than the third may be ſolved by means of the dividers of the known term; but, from the greater number of terms on one ſide of the equation, the proceſs will be often very tedious. The order of the equation is determined by the higheſt power of the unknown number; if it is the fourth power, the equation is of the fourth order; if it is the fifth power, the equation is of the fifth order; and if n is the index of the higheſt power of the unknown number, the equation is of the n^{th} order. An equation of the n^{th} order may be repreſented generally in the following manner:

$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + dx^{n-4} \ldots mx = l$;

and, as ſome of theſe terms, inſtead of being added, may be taken away, by changing the marks of thoſe terms, this equation may be changed into any of the forms of the n^{th} order.

Since $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \ldots mx = l$,

$x \times \overline{x^{n-1} + ax^{n-2} + bx^{n-3} + cx^{n-4} \ldots m} = l.$

Therefore, among the dividers of l one will be found equal to x, whoſe co-part is equal to

$$x^{n-1} + ax^{n-2} + bx^{n-3} + cx^{n-4} \ldots m.$$

To find with greater eaſe the divider which ſolves the equation, inſtead of three ſeries, five may be uſed often with great advantage; and as the upper and lower ſeries were before made by ſubſtituting the unknown number for two others, the one greater and the other leſs by unity,

unity, the two additional ſeries are made by ſubſtituting for the unknown number two others, the one greater and the other leſs by two. Conſequently, if among the dividers of the known ſide of the given equation one is found differing from ſome of the dividers of the new equations by one and two, the number thus found will ſolve the given equation.

Let $x^n + ax^{n-1} + bx = l$.

Make $x = y + 2$

$$\left.\begin{array}{r} \therefore y^n + ny^{n-1}2 \ldots\ldots\ldots + 2^n \\ + ay^{n-1} \ldots + a2^{n-1} \\ + by + b.2 \end{array}\right\} = l$$

$$\left.\begin{array}{r} \therefore y^n + ny^{n-1}2 \ldots\ldots \\ + ay^{n-1} \ldots \\ + by \end{array}\right\} = l - 2^n - a2^{n-1} - 2b$$

If $x = y - 2$.

$$\left.\begin{array}{r} y^n - ny^{n-1}2 + \ldots\ldots\ldots \\ + ay^{n-1} \\ + by - \end{array}\right\} l \pm 2^n \mp a2^{n-1} \mp 2b$$

Let $y^4 + 14y^3 + 9y^2 = 1296$.

$$\left.\begin{array}{l} 1296 - 2^4 - 14.2^3 - 9.2^2 = 1132 \\ 1296 - 1 - 14 - 9 = 1272 \\ \qquad\qquad\qquad\quad = 1296 \\ 1296 - 1 + 14 - 9 = 1300 \\ 1296 - 2^4 + 14.2^3 - 9.2^2 = 1356 \end{array}\right| \begin{array}{l} 1.2.4 \\ 1.2.3.4 \\ 1.2.3.4. \\ 1.2.4.5 \\ 1.2.3.4.6. \end{array}$$

y is leſs than $\sqrt[4]{1296}$: $\therefore$ the dividers equal to, or above 6, need not be tried.

There

There are two dividers of 1296, differing from the dividers in the neareſt ſeries above and below them by one, namely, 3 and 4. Therefore, if we had no other data, theſe two numbers muſt have been tried in the co-part of x. But though 3 is greater than a divider in the uppermoſt ſeries by two, yet there is no divider in the loweſt ſeries greater than it by two: and conſequently three cannot be the required divider: 4 is, and its co-part $324 = y^3 + 14y^2 + 9y$.

If the unknown number is large, which may be often ſeen from comparing together the known ſide with a few terms on the other ſide of the equation, it will be uſeful to change the given equation into one in which the unknown number ſhall be a given part of the unknown number in the propoſed equation. Thus, in the equation

$$x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ ----- } mx = l.$$

$$\text{Let } y = \frac{x}{p}. \quad \therefore x = py.$$

$$\therefore p^n y^n + ap^{n-1} y^{n-1} + bp^{n-2} y^{n-2} + cp^{n-3} y^{n-3} \text{ --- } mpy = l.$$

Divide both ſides by p^n.

$$\therefore y^n + \frac{ay^{n-1}}{p} + \frac{by^{n-2}}{p^2} + \frac{cy^{n-3}}{p^3} \text{ --- } \frac{my}{p^{n-1}} = \frac{l}{p^n}.$$

Thus the known ſide may be made leſs, and the number of dividers may be decreaſed. This change is made at once by dividing the ſecond term of the propoſed equation by the number denoting the part which the new unknown number is of the other, and the terms in ſucceſſion by

This change is of uſe, when the known ſide of the equation is a fraction; as it may be made a whole number if p is made large enough.

$$\text{Let } x^4 - 2x^3 + 8x^2 - 3x = \frac{5}{16}.$$

$$\text{Make } x = \frac{y}{2}$$

$$\therefore y^4 - 4y^3 + 32y^2 - 24y = \frac{5}{16} \times 2^4 = 5$$

$$\therefore y = 1$$

$$\therefore x = \tfrac{1}{2}.$$

If x had been made equal to $\frac{y}{4}$, the known ſide would have been $\frac{5}{16} \times 4^4 = \frac{5 \times 4^4}{4^2} = 5 \times 4^2 = 80$, and y would have been found among the dividers of 80.

When in the propoſed equation there are ſeveral terms to be taken away, it is uſeful to find ſuch a value for the unknown number, as ſhall make the difference between the terms to be added and taken away equal to nothing, and thus a limit to the unknown number is found.

$$\text{Let } x^5 - 2x^3 = 896.$$

If $x^5 - 2x^3 = 0$: $x^3 \times \overline{x^2 - 2} = 0$, and $x = \sqrt{2}$. Conſequently x muſt be greater than $\sqrt{2}$. Alſo, ſince $x^3 \times \overline{x^2 - 2} = 896$, x muſt be leſs than the third root of 896, or leſs than 10.

Let

Let $x^5 - 12x^3 = 256$:

$\therefore x^3 \times \overline{x^2 - 12} = 256$

$\therefore x$ muſt be leſs than $\sqrt[3]{256}$ or leſs than 7, and if $x^2 - 12 = 0$, $x = \sqrt{12}$. Therefore x muſt be greater than 3.

Let $x^5 - 6x^4 + 9x^3 - 2x^2 + 12x = 24$:

$\therefore x \times \overline{x^4 + 9x^2 + 12 - x \times \overline{6x^2 + 2}} = 24.$

If $x^4 + 9x^2 + 12 = 6x^3 + 2x$, one ſide of the equation is equal to 0; and in that caſe

$6x^3 + 2x - x^4 - 9x^2 = 12;$

or $x \times \overline{6x^2 + 2} - x \times \overline{x^3 + 9x} = 12$

$\therefore 6x^2 + 2 < x^3 + 9x.$

But if $6x^2 + 2 = x^3 + 9x$,

then $6x^2 + 2 = x \times \overline{x^2 + 9}$

$\therefore 6x + \frac{2}{x} = x + 9$

$\therefore 6x = x + 9$ nearly

$\therefore 6x - x = 9$ nearly

$\therefore x \times \overline{6 - x} = 9$ nearly

$\therefore x = 3$ nearly.

Hence x muſt be leſs than three. Let x be made equal to two, and conſequently its co-part $12 = x^4 + 9x^2 + 12 - 6x^3 - 2x$:

$\therefore x^3 + 9x = 6x^2 + 2$

$\therefore 8 + 18 = 6.4 + 2.$

Hence 2 is a value of the unknown number.

Now,

Now, if x is made ſucceſſively greater than two, its co-part diminiſhes, and then increaſes; the product is firſt leſs, and afterward greater, than 24, and conſequently there is another number which, being ſubſtituted for x, will ſolve the equation.

$$\text{Let } x^4 - 4x^3 - 5x^2 + 36x = 36:$$

$$\therefore x \times \overline{x^3 - 4x^2 - 5x + 36} = 36.$$

$$\text{If } x^3 + 36 = 4x^2 + 5x,$$

$$x \times \overline{4x + 5 - x^2} = 36.$$

x muſt be greater than one, and conſequently in the propoſed equation x muſt be greater than one. And if x were equal to 4, $36 - 5x = 9$. But $36 - 5x$, in that caſe, is greater than 9. $\therefore x$ is leſs than 4. There are two dividers of 36 lying between the limits thus found, namely, 2 and 3. Conſequently, if $x = 2$, $x^3 - 4x^2 - 5x + 36 = 18$:

$$\text{Or } x \times \overline{4x + 5 - x^2} = 18,$$

which it does, and conſequently $x = 2$.

$$\text{Or if } x = 3,$$

$$\text{then } x^3 - 4x^2 - 5x + 36 = 12.$$

$$\therefore x \times \overline{4x + 5 - x^2} = 24$$

$$\therefore 4x + 5 - x^2 = 8,$$

which it does, and conſequently $x = 3$.

Or

Or this might have been done by the three ſeries of dividers.

$$36 - 1 + 4 + 5 - 36 = 8 \left\{ \begin{matrix} 1\ 2\ 4\ 8 \\ 36 \end{matrix} \right\} \begin{matrix} 1\ 2\ 3\ 4\ 6\ 9\ 12\ 18\ 36 \\ 1\ 2\ 3\ 4\ 6\ 8\ 9\ 12\ 18\ 24 \end{matrix}$$
$$36 - 1 - 4 + 5 + 36 = 72$$

2 and 3 are the only numbers which can ſolve the given equation.

$$\text{Let } x^4 + 3x^3 - 7x^2 - 27x = 18$$
$$\therefore x \times \overline{x^3 + 3x^2 - 7x - 27} = 18.$$

If $x^3 + 3x^2 = 7x + 27$,

$$x \times \overline{x^2 + 3x - 7} = 27.$$

$\therefore x < 2$ and $x > 3$ in the limiting equation.

If $x = 3$, $x^3 + 3x^2 - 7x - 27 = 6$, and this is found to ſucceed; $\therefore x = 3$. By the mode of dividers the ſame will be found.

$$18 - 1 - 3 + 7 + 27 = 48 \left\{ \begin{matrix} 1\ 2\ 3\ 4\ 6\ 8 \\ 18 \end{matrix} \right\} \begin{matrix} 1\ 2\ 3\ 6\ 9 \\ \end{matrix}$$
$$18 - 1 + 3 + 7 - 27 = 0$$

The only dividers of 18, which have a number leſs by unity in the upper ſeries, are 2, 3, and 9, and conſequently if 3, 4, and 10 had been found in the lower ſeries, the former numbers 2, 3, and 9 would have ſolved the equation. But on attempting to take dividers of the lower ſeries, it is found, that, by ſubſtituting $y - 1$ for x, or writing down the terms to be taken away from 18, there is no

number from which dividers can be taken in that form. But, on examining the new equation, a number is found, by which the value of the unknown number may be diſcovered.

Let $x = y - 1$.

$$\left.\begin{array}{r} y^4 - 4y^3 + 6y^2 - 4y + 1 \\ + 3y^3 - 9y^2 + 9y - 3 \\ - 7y^2 + 14y - 7 \\ - 27y + 27 \end{array}\right\} = 18$$

$$\therefore y^4 - y^3 - 10y^2 - 8y = 0.$$

$$\therefore y^3 - y^2 - 10y = 8$$

$$\therefore y \times \overline{y^2 - y - 10} = 8$$

$$\therefore y = 4.$$

And 4 is greater than x by unity. Conſequently, if among the dividers of 18 there is a number leſs than 4 by unity, that will ſolve the propoſed equation, and no other number can. On ſearching among the dividers of 18 three is found, which therefore ſolves the equation.

Let $x^4 - 6x^3 + 5x^2 + 2x = 10$.

$$\therefore x \times \overline{x^3 - 6x^2 + 5x + 2} = 10$$

$$\left.\begin{array}{l} 10 - 1 + 6 - 5 - 2 = 8 \\ \qquad\qquad\qquad\quad 10 \\ 10 - 1 - 6 - 5 + 2 = 0 \end{array}\right\} \begin{array}{l} 1\ 2\ 3\ 4 \\ 1\ 2\ 5 \end{array}$$

Let

Let $x = y - 1$.

$$\left.\begin{array}{r} y^4 - 4y^3 + 6y^2 - 4y + 1 \\ - 6y^3 + 18y^2 - 18y + 6 \\ + 5y^2 - 10y + 5 \\ + 2y - 2 \end{array}\right\} = 10$$

$$\therefore y^4 - 10y^3 + 29y^2 - 30y = 0$$

$$\therefore y^3 - 10y^2 + 29y = 30$$

$$\therefore y \times \overline{y^2 - 10y + 29} = 30$$

$$\therefore y = 6.$$

Among the dividers of 10, there is a number leſs than 6 by unity, conſequently 5 ſolves the propoſed equation.

Hence, if in attempting to ſolve an equation by means of three ſeries of dividers, it ſhould appear that the lower ſeries is wanting, the equation may be reduced to one of an inferior order with an unknown number greater than the unknown number in the given equation by unit.

In the higher equations, as in thoſe of the third order, an approximation may be made to the unknown number by changing it into another, greater or leſs by a given difference.

Let $x^4 + 12x = 17$.

x is evidently greater than one, and leſs than 2.

Let $x = y + 1$.

$$\therefore \left.\begin{array}{r} y^4 + 4y^3 + 6y^2 + 4y + 1 \\ + 12y + 12 \end{array}\right\} = 17$$

$\therefore$

$$\therefore y^4 + 4y^3 + 6y^2 + 16y = 4$$

$$\therefore y \times \overline{y^3 + 4y^2 + 6y + 16} = 4$$

$$\left.\begin{array}{l} ,1 : ,2 : ,3 : ,4 : ,5 : ,6 \\ 40 : 20 : 13 : 10 : 8 : 6,6 \end{array}\right\} \begin{array}{l} \text{Dividers.} \\ \text{Co-parts.} \end{array}$$

$y^3 + 4y^2 + 6y + 16$ is greater than 16, and leſs than 20. Therefore y is between ,2 and ,3.

$$\therefore x = 1,3 + z.$$

Subſtitute $1,3 + z$ for x, and from the new equation another figure may be obtained, and ſo on.

Before an attempt is made to ſolve an equation of a very high order, it may be uſeful to examine, whether it may not be depreſſed into one of an inferiour order whoſe ſolution is eaſier. Thus, let $x^6 - ax^4 + bx^2 = l$ be propoſed. Then by making $y = x^2$, a new equation is formed, $y^3 - ay^2 + by = l$, from the ſolution of which x is determined by finding the ſecond root of y.

$$\text{If } x^8 - ax^4 = l.$$

$$\text{Let } y = x^4$$

$$\therefore y^2 - ay = l$$

and $x =$ the fourth root of the unknown number in the laſt equation.

In general an equation may be depreſſed to one of a lower order, if the indexes of the unknown number in the given equation are diviſible by the ſame number. In the equation $x^6 - ax^4 + bx^2 = l$, the indexes 6, 4, 2 are diviſible

diviſible by 2; but the equation $x^5 - ax^3 + bx^2$ cannot be depreſſed, becauſe 5, 3, 2 are not diviſible by the ſame number.

ON THE MODE OF SUMMING A SERIES OF NUMBERS

WHICH FOLLOW A CERTAIN ORDER.

IN endeavouring to find a decimal equal to a common fraction, it has been ſhewn that a ſeries of figures ſometimes recurs, and that any given number of theſe figures will differ from the common fraction. The diviſion, which produces this decimal, may be continued without end, and hence the decimal is called an infinite ſeries. The ſum of ſuch a ſeries cannot be found; but the limit to which it is continually approaching, may be aſcertained. $\frac{1}{3} = ,33333 - - - -$ or $\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} - - - - -$ The law of this ſeries is, that each term is one-tenth part of the preceding term. In the ſame manner other ſeries are formed; and from the law which is followed in the formation of them they acquire peculiar names.

If in a ſeries of numbers each term differs from the preceding term by a given number, the terms are ſaid to be in arithmetical progreſſion. Thus, 1, 2, 3, 4, 5 is a ſeries of numbers in arithmetical progreſſion, each number exceeding the preceding number by unity. 8, 6, 4, 2

is alſo a ſeries in a ſimilar progreſſion but decreaſing, each term being leſs than the preceding term by 2,

Let A B C D - - - - - - - - K L repreſent any ſeries of numbers in arithmetical progreſſion, whoſe firſt term is A and laſt term L. Let n = the number of terms, and b the common difference. If the numbers increaſe, b is to be added to any term to make the ſubſequent term; if the numbers decreaſe, b is to be taken away from any term to make the ſubſequent term.

Let the numbers progreſſively increaſe; then, ſince A is the firſt term, $A + b$ is the ſecond term $= B$; and the third term $C = B + b = A + 2b$; the fourth term $D = C + b = A + 3b$, and ſo on. Conſequently ſince b is added to A for the ſecond term, and is conſtantly added for each term afterwards, any term in the ſeries may be found without knowing the preceding term. If the fourth term is required, b muſt have been added three times to make it. Therefore the fourth term is $A + 3b$. The fifth term is $A + 4b$. The ſixth $A + 5b$; and if $m =$ any number, the m^{th} term $= A + \overline{m-1} \cdot b$. Conſequently if $m = n$. $L = A + \overline{n-1} \cdot b$.

Hence, if of the four things, the firſt term, the laſt term, the number of terms, and the common difference, any three are given, the fourth may be found by means of a ſimple equation. If the laſt term, the number of terms, and the common difference are given, the firſt term $A = L - b \times \overline{n-1}$. If the firſt term, laſt term, and number of terms are given, the common difference $b = \frac{L - A}{n-1}$.

If

If the firſt term, laſt term, and common difference are given, the number of terms $n = \frac{L - A + b}{b}$.

What is the laſt term of the ſeries 2, 4, 6, 8 - - - - - - - having twenty terms? Here $A = 2$, $b = 2$, and $n = 20$.

$$L = A + \overline{n-1} \cdot b = 2 + \overline{20 - 1} \cdot 2 = 2 + 38 = 40.$$

What is the firſt term of the ſeries - - - - - - 54, 57, 60 having 10 terms, and 60 being the laſt term? There are three terms of this ſeries written down, conſequently the common difference is given, for it is equal to 57 — 54, or 3; and

$$A = L - b \times \overline{n-1} = 60 - 3 \times \overline{10-1} = 60 - 27 = 33.$$

What is the common difference in the ſeries 27 - - - - - - - - - - - - 71, the firſt and laſt terms only being known, and there being twelve terms?

$$b = \frac{L - A}{n - 1} = \frac{71 - 27}{12 - 1} = \frac{44}{11} = 4.$$

What is the number of terms in the ſeries 12, 20, 28 - - - - - - - - - - - - - - - 92?

$$n = \frac{L - A + b}{b} = \frac{92 - 12 + 8}{8} = \frac{88}{8} = 11.$$

In a ſeries of numbers in arithmetical progreſſion the ſum of any two numbers, equally diſtant from the extremes, is equal to the ſum of the firſt and laſt terms.

Let A B C D E - - - - - - I K L repreſent a ſeries of numbers in arithmetical progreſſion. Then A + L = ſum of firſt and laſt terms. The ſecond term B = A + b, and the laſt term but one K = L − b.

$$\therefore B + K = A + b + L - b = A + L.$$

The third term C = A + 2b, and the laſt term but two, I = L − 2b.

$$\therefore C + I = A + 2b + L - 2b = A + L.$$

In the ſame manner it will appear, that any other two numbers equally diſtant from the extremes = A + L.

If there is an odd number of terms in the ſeries, the middle term is equal to half the ſum of the firſt and laſt terms.

For the middle term is the term denoted by $\frac{n-1}{2} + 1$ or $\frac{n+1}{2} = m$. $\therefore \overline{m-1} \cdot b = \overline{\frac{n+1}{2} - 1} \cdot b = \frac{n-1}{2} \cdot b$. Therefore the middle term $= A + \frac{n-1}{2} \cdot b = \frac{A + A + \overline{n-1} \cdot b}{2}$. But $A + \overline{n-1} \cdot b = L$. $\therefore$ the middle term $= \frac{A + L}{2}$

The ſum of a ſeries of numbers in arithmetical progreſſion is equal to the ſum of the firſt and laſt terms multiplied into half the number of terms.

Let

Let the number of terms be even : then the ſum of the ſeries is equal to the ſum of the firſt and laſt terms, and of all the terms equally diſtant from them : and ſince every pair of terms thus taken is equal to A + L, and there will be as many pairs as units in half the number of terms, the ſum muſt be equal to $\overline{A+L} \times \frac{n}{2}$.

If the number of terms is odd, the ſum is equal to the ſum of the middle term, and all the pairs of terms equally diſtant from it; and conſequently it is equal to $\frac{A+L}{2} + \frac{n-1}{2} \times \overline{A+L} = \frac{A+L+nA-A+nL-L}{2} = \frac{nA+nL}{2} = \overline{A+L} \times \frac{n}{2}$.

Hence, if S = the ſum of the ſeries, ſince A, L and n may be found in terms of each other, of the five things A, L, n, b and S, any three being given, the other two may be found.

$$S = \overline{A+L} \times \frac{n}{2} = \overline{L - b \times \overline{n-1} + L} \times \frac{n}{2} = \frac{2L - b \times \overline{n-1}}{2} \times n.$$

$$S = \overline{A+L} \times \frac{n}{2} = \overline{A + A + \overline{n-1}.b} \times \frac{n}{2} = \frac{2A + \overline{n-1}.b}{2} \times n.$$

From the firſt equation

$$A = \frac{2S}{n} - L = \frac{2S - nL}{n};$$

And

And from the ſecond

$$2A = \frac{2S}{n} - \overline{n-1}.b. \quad \therefore A = \frac{2S - n.\overline{n-1}.b}{2.n}.$$

From the firſt equation

$$L = \frac{2S}{n} - A = \frac{2S - nA}{n}; \text{ and}$$

$$2L = \frac{2S}{n} + b \times \overline{n-1}. \quad \therefore L = \frac{S}{n} + b \times \frac{n-1}{2}.$$

From the firſt equation

$$b = \overline{2L - \frac{2S}{n}} \times \frac{1}{n-1} = \frac{2}{n-1} \times \frac{nL - S}{n};$$

And from the ſecond

$$b = \overline{\frac{2S}{n} - 2A} \times \frac{1}{n-1} = \frac{2}{n-1} \times \frac{S - nA}{n}.$$

From the firſt equation

$$n = \frac{2S}{A + L}, \text{ and}$$

$$2S = 2nL - n^2b + nb.$$

$$\therefore n \times \overline{2L + b - nb} = 2S.$$

If the terms of the ſeries decreaſe, the ſame equations may be uſed, by changing only A for L and L for A. For by making the laſt term of a decreaſing ſeries the firſt term of the ſame ſeries, read the contrary way, a ſeries of terms, increaſing by a common difference is formed.

$a, a + 2b, a + 3b$ - - - - $a + \overline{n-2}, a + \overline{n-1}\,b$ is the same as $a + \overline{n-1}b, a + \overline{n-2}b, a + \overline{n-3}.b$ - - - - - $a + 2b, a$; in the latter case $a + \overline{n-1}.b$ being the first term, and a the last term: but in the former case a was the first term, and $a + \overline{n-1}.b$ the last term.

What is the sum and number of terms of the series 8, 12, 16 - - - - - - - - - - - - 96?

In this series A = 8, L = 96, and $b = 4$.

$$L = A + \overline{n-1}.b. \quad \therefore 96 = 8 + \overline{n-1}.4 = 4 + 4n$$

$$\therefore n = \frac{96-4}{4} = \frac{92}{4} = 23$$

$$\therefore S = \overline{A \times L} \times \frac{n}{2} = \overline{8+96} \times \frac{23}{2} = 1196.$$

What are the first and last terms of a series in arithmetical progression, whose sum is 1196, common difference 4, and number of terms 23?

$$A = \frac{S}{n} - \frac{n-1}{2}.b = \frac{1196}{23} - 11.4 = 52 - 44 = 8.$$

$$L = A + \overline{n-1}.b = 8 + 22.4 = 96.$$

What is the number of terms in a series whose first term is 8, sum 1196, and common difference 4?

$$S = \overline{A+L} \times \frac{n}{2}$$

$$\therefore \frac{2S}{n} = A + A + \overline{n-1}.b = 2A + \overline{n-1}.b$$

$\therefore$

$$\therefore 2S = 2An + n^2b - nb = bn^2 + n \cdot \overline{2A - b}$$

$$\therefore \frac{2S}{b} = n^2 + n \cdot \frac{2A - b}{b}$$

$$\therefore \frac{2 \cdot 1196}{4} = n^2 + n \cdot \frac{16 - 4}{4} = n^2 + 3n = 598$$

$$\therefore n = \sqrt{598 + \frac{9}{4}} - \frac{3}{2} = \frac{49}{2} - \frac{3}{2} = 23.$$

What is the common difference in a ſeries whoſe firſt term is 8, ſum 1196, and number of terms 23?

$$S = \overline{A + A + \overline{n-1} \cdot b} \times \frac{n}{2}$$

$$\therefore 1196 = \overline{8 + 8 + 22 \cdot b} \times \frac{23}{2}$$

$$\therefore \frac{1196}{23} = 8 + 11b$$

$$\therefore 11b = \frac{1196}{23} - 8 = \frac{1196 - 184}{23} = \frac{1012}{23} = 44$$

$$\therefore b = 4.$$

What is the ſum of twenty terms of the ſeries 50, 46, 42 -------?

$$b = 50 - 46 = 4$$

$$A = 50$$

$$\therefore L = A - b \cdot \overline{n-1} = 50 - 4 \times 19 = 50 - 76.$$

The queſtion involves an abſurdity, for 76 is greater than 50, and conſequently there cannot be 20 terms in this ſeries.

How

How many terms can there be in the ſeries 50, 46, 42 - - - - - - - - - - - ?

Since the common difference is 4, and the terms are even numbers, the laſt term muſt be either two or four. If four is taken for the laſt term.

$$4 = 50 - 4 \cdot \overline{n - 1}$$

$$\therefore n - 1 \cdot 4 = 50 - 4 = 46$$

$$\therefore n - 1 = \frac{46}{4} = 11\tfrac{1}{2}$$

But the number of terms muſt be a whole number, therefore b is not equal to 4, but to 2; and

$$2 = 50 - 2 \cdot n - 1$$

$$\therefore n - 1 = \frac{50 - 2}{2} = \frac{48}{2} = 24$$

$$\therefore n = 25.$$

If in a ſeries of numbers each term is a given part or multiple of the preceding term, the numbers are ſaid to be in geometrical progreſſion.

Let A B C D - - - - - - - - - - I, K, L repreſent a ſeries of numbers in geometrical progreſſion, and let r denote the given part or multiple. Then, if the numbers are in an increaſing progreſſion, $B = rA$, $C = rB = r^2A$, $D = rC = r^3A$. And the ſeries may be written down thus:

$$A \times \overline{1 \; r \; r^2 \; r^3 \; - - - - - - - - - - - - - \; r^{n-2} \; r^{n-1}}.$$

The

The laſt term is r, raiſed to the index $n - 1$, becauſe it begins with the ſecond term, and is continually multiplied into itſelf to produce the ſubſequent terms. $\therefore L = A \times r^{n-1}$: and if m repreſents the number of any term in the ſeries, the term itſelf is equal to $A \times r^{m-1}$.

In a geometrical progreſſion, the product of any two terms equally diſtant from the extremes is equal to the product of the firſt and laſt terms.

$$B = A \times r \qquad K = A \times r^{n-2}$$
$$\therefore B \times K = A \times r \times A \times r^{n-2} = A \times A r^{n-1} = A \times L$$
$$C = A \times r^2 \qquad I = A \times r^{n-3}$$
$$\therefore C \times I = A \times r^2 \times A \times r^{n-3} = A \times A \times r^{n-1} = A \times L.$$

If the number of terms is odd, the middle term is equal to the ſecond root of the product of the firſt and laſt terms.

$$\text{Let } M = \text{middle term, then } m = \frac{n+1}{2}.$$

$$M = A \times r^{m-1} = A \times r^{\frac{n+1}{2}-1} = A \times r^{\frac{n-1}{2}}$$
$$\therefore M^2 = A^2 \times r^{n-1} = A \times A \times r^{n-1} = A \times L$$
$$\therefore M = \sqrt[2]{A \times L}.$$

In finding from three out of the four things given A, L, r, and n the fourth, not only equations of the higher orders are ſometimes involved, but ſome which cannot be ſolved by any principles hitherto explained. If the firſt term, and common multiplier, and number of terms

terms are given, the term will be found by the following equation.

$$L = Ar^{n-1}.$$

If the firſt term, laſt term, and number of terms are given, the common multiplier may be found by the equation $r^{n-1} = \frac{L}{A}$, which is an equation of the ſecond or higher orders.

$A = \frac{L}{r^{n-1}}$, conſequently L, r, and n being given, A may be found.

$r^{n-1} = \frac{L}{A}$, conſequently if A, L, and r are given in this equation, the index of r is unknown, and this is to be diſcovered by the means of logarithms, which are themſelves formed from the indexes of a ſeries of numbers in geometrical progreſſion.

Let $1\ r\ r^2\ r^3 \ldots\ldots\ldots\ldots r^{n-1}$ repreſent a ſeries of numbers in geometrical progreſſion, and let another ſeries of numbers be ſo conſtituted, that each term may correſpond to a term in the geometrical ſeries, and be equal to the index of r multiplied into ſome given number. Thus, let the index of the terms be multiplied in ſucceſſion by the number 4, and the two ſeries may be written down, the laſt formed ſeries under the other. Thus,

$$1\ r\ r^2\ r^3 \ldots\ldots\ldots\ldots r^{n-2}\ r^{n-1}$$

$$0\ 4\ 8\ 12 \ldots\ldots\ldots\ldots 4.\overline{n-2}\ 4\overline{n-1}.$$

The

The lower ſeries is called a ſeries of logarithms, and the terms in the lower ſeries are called the logarithms of the terms in the upper ſeries, to which they correſpond.

If r is made equal to ten; and inſtead of four for the multiple of the indexes, one is uſed, the two ſeries become

1	10	100	1000	10000	-----	10^{n-2}	10^{n-1}
0	1	2	3	4	-----	$n-2$	$n-1$.

Thus 1 is the logarithm of 10, 2 of 100, 3 of 1000, and ſo on. Between the numbers 1 and 10, 10 and 100, 100 and 1000, a ſeries may be inſerted of other numbers in geometrical progreſſion. Thus,

$$p \quad p^2 \quad p^3 \quad p^4 \text{ -------- } p^{n-1}$$

Whoſe laſt term $p^{n-1} = 10$.

In the ſame manner between 10 and 100 a ſimilar ſeries may be inſerted.

$$q \quad q^2 \quad q^3 \quad q^4 \text{ -------- } q^{n-1};$$

q^{n-1} being equal to 100.

Conſequently a ſeries of logarithms may be formed for numbers between 1 and 10, 10 and 100, and the other numbers.

Suppoſe that there was only one term between 1 and 10, which may be called x.

$$\therefore 1 : x :: x : 10. \quad \therefore x^2 = 10. \quad \therefore x = 10^{\frac{1}{2}}.$$

Between

Between 1 and $10^{\frac{1}{2}}$, in the ſame manner, let there be only one term y. $\therefore 1 : y :: y : 10^{\frac{1}{2}}$.

$$\therefore y^2 = 10^{\frac{1}{2}}. \quad \therefore y = 10^{\frac{1}{4}}.$$

Between 1 and $10^{\frac{1}{4}}$ another term may be inſerted, which will be equal to $10^{\frac{1}{8}}$; and again another number may be in like manner found equal to $10^{\frac{1}{16}}$. Thus between 1 and 10 a ſeries of numbers will be formed.

$$1,\ 10^{\frac{1}{32}},\ 10^{\frac{1}{16}},\ 10^{\frac{1}{8}},\ 10^{\frac{1}{4}},\ 10^{\frac{1}{2}},\ 10,$$

whoſe logarithms will be

$$0,\ \tfrac{1}{32},\ \tfrac{1}{16},\ \tfrac{1}{8},\ \tfrac{1}{4},\ \tfrac{1}{2},\ 1.$$

Between any two numbers in theſe ſeries may be inſerted other numbers without end, for there is no limit to the numbers between 1 and 10, 1 and $10^{\frac{1}{2}}$, or $10^{\frac{1}{32}}$ and $10^{\frac{1}{16}}$.

Hence, if the logarithm of a number is given, the number may be found.

Let the logarithm $= \frac{1}{2}$.

$\frac{1}{2}$ is leſs than 1. $\therefore$ the number is leſs than 10, and it equals $10^{\frac{1}{2}} = 3{,}1622$ - - - - -

Let the logarithm $= \frac{1}{4}$.

Then the number is $10^{\frac{1}{4}}$, or $\sqrt[4]{3{,}1622}$

If the logarithm is between 1 and 2, the number is between 10 and 100. And, as before, it is found by making the logarithm the index of the power of 10. Let the logarithm $= \frac{3}{2}$, then the number is $10^{\frac{3}{2}}$, or $\sqrt[2]{1000} = 33{,}333\text{-----}$

If the logarithm is $\frac{7}{3}$, the number is greater than 100, and is equal to $10^{\frac{7}{3}}$.

Thus, as every number may be repreſented by 10 raiſed to ſome power, every number will have its correſponding logarithm; and tables may be formed, upon theſe principles, of numbers and their logarithms. It would indeed be a laborious work to find every number, correſponding to every poſſible index of ten, by the rules given for finding the roots of numbers; and hence many methods have been deviſed to render the proceſs more eaſy. Theſe may be found in the books treating particularly on logarithms; but it is ſufficient for the preſent purpoſe, that the learner ſhould underſtand a mode of forming a ſeries of numbers, which are the logarithms of another ſeries of numbers.

Logarithms are of great uſe in all problems which involve the multiplication or diviſion of large numbers; as theſe operations may be performed with tables of logarithms, by means of addition or ſubtraction.

Let a be the logarithm of 10^a

b 10^b

Then

Then $10^a \times 10^b = 10^{a+b}$, and the logarithm of 10^{a+b} is $a+b$, or the ſum of the logarithms of 10^a and 10^b. Conſequently, if two numbers are to be multiplied together, by taking from the tables their logarithms and adding them together, the ſum will be the logarithm of the product of thoſe numbers: and by looking into the tables for the logarithm correſponding to the ſum thus found, the number oppoſite to it is the number equal to the product of the two numbers to be multiplied together.

If a number is to be divided by another number, the reſult may be found by looking into the tables for the number correſponding to a logarithm, which is equal to the difference of the logarithms of the two numbers.

$$10^a \div 10^b = 10^{a-b}.$$

$\therefore$ The log. of $\frac{10^a}{10^b}$ = log. of $10^{a-b} = a - b$ = the difference of the logarithms of 10^a and 10^b. And the number, correſponding to $a-b$ in the tables, is the number which would ariſe from dividing 10^a by 10^b.

Let 10^a, as before, repreſent any number; then the n^{th} power of $10^a = 10^{na}$, and the log. of $10^{na} = na$, or n times the log. of 10^a. Hence any power of a number may be eaſily found by the logarithmic tables, by multiplying the logarithm of the number into the index of the power to which it is to be raiſed, and taking the number correſponding to the logarithm thus found out of the tables.

In the ſame manner the root of a number may be found by the tables. Let 10^a repreſent the number, and n the

index of the root required. Then $10^{\frac{a}{n}}$ = the n^{th} root of the number: but the logarithm of $10^{\frac{a}{n}} = \frac{a}{n}$ Divide therefore the logarithm of the given number by the index of the root required, and the quotient gives a logarithm whose corresponding number is the root required.

If the power of a number is known, though the index itself is an unknown number, the index may be found. Let $a^x = b$. Therefore the logarithm of a^x = the logarithm of b, but the logarithm of $a^x = x$ log. of a.

$$\therefore x = \frac{\text{log. of } b}{\text{log. of } a}.$$

Hence a method is derived of discovering the number of terms in a geometrical progression, when the first and last terms and the common multiplier are given. For $r^{n-1} = \frac{L}{A}$ (see page 175).

$$\therefore \overline{n-1} \cdot \text{log. } r = \text{log. L} - \text{log. A}$$

$$\therefore n - 1 = \frac{\text{log. L} - \text{log. A}}{\text{log. } r}$$

$$\therefore n = \frac{\text{log. L} - \text{log. A} + \text{log. } r}{\text{log. } r}.$$

In a series of numbers in geometrical progression, the first term is 100, the last 1000000000000, and the common multiplier 10. What is the number of terms?

A =

$$A = 100 = 10^2. \quad \therefore \text{Log. } A = 2$$

$$L = 1000000000000 = 10^{12}. \quad \therefore \text{Log. } L = 12.$$

$$r = 10. \quad \therefore \text{Log. } r = 1$$

$$\therefore n = \frac{12 - 2 + 1}{1} = 11.$$

In this caſe the number n is eaſily found, but in general the tables muſt be conſulted for the logarithms of A, L, and r.

The firſt term of a geometrical progreſſion is four, the common multiplier 2, and the number of terms 10, what is the laſt term?

$$L = Ar^{n-1} = 4 \times 2^9 = 4 \times 512 = 2048.$$

If the firſt term had been 12, the common multiplier 18, and the number of terms 50, the laſt term could not have been found without great trouble by the above method, ſince 18 muſt have been raiſed to the 49th power. In this and ſimilar caſes, therefore, logarithms ſhould be uſed.

$$L = Ar^{n-1}$$

$$\therefore \text{Log. of } L = \text{log. of } Ar^{n-1}.$$

$$\text{But log. of } Ar^{n-1} = \text{log. of } A + \text{log. } r^{n-1}$$

$$\text{And log. of } r^{n-1} = \overline{n-1} \cdot \times \text{log. of } r$$

$$\therefore \text{Log. of } L = \text{log. of } A + \overline{n-1} \cdot \text{log. of } r$$

$$\therefore \text{Log. of } L = \text{log. of } 12 + 49 \times \text{log. of } 18.$$

Find in the tables the logarithms of 12 and 18; multiply the latter logarithm by 49, and add the product to the

logarithm of 12. The ſum is equal to the logarithm of L, and conſequently the number correſponding to this logarithm in the tables, is equal to L.

What is the middle term of 11 numbers in geometrical progreſſion, whoſe firſt term is 100, and laſt term 100000000000?

$$M = \sqrt[2]{A \times L}$$

$$\therefore \text{Log. of } M = \frac{\text{log. of } A + \text{log. of } L}{2}$$

$$\therefore \text{Log. of } M = \frac{2 + 12}{2} = \frac{14}{2} = 7$$

$$\therefore M = 10^7 = 10000000.$$

What is the middle term of 9 numbers, in geometrical progreſſion, whoſe firſt term is 4 and common multiplier 3?

$$L = Ar^{n-1} = 4 \cdot 3^8 = 4 \cdot 6561 = 26244.$$

$$M = \sqrt{A \times L} = \sqrt{4 \times 26244} = 2\sqrt{26244} = 2 \times 162 = 324.$$

What is the common multiplier in a progreſſion, whoſe firſt term is 100, laſt term 10000000, and number of terms 6?

$$r^{n-1} = \frac{L}{A}$$

$$\therefore r^5 = \frac{10000000}{100} = 100000$$

$$\therefore 5 \text{ log. of } r = \text{log. of } 100000 = 5$$

$$\therefore \text{Log. of } r = 1$$

$$\therefore r = 10.$$

What

What is the common multiplier in a progreſſion, whoſe firſt term is 9, laſt term 9216, and number of terms 6?

$$r^{n-1} = \frac{L}{A}$$

$$\therefore r^5 = \frac{9216}{9} = 1024$$

$$\therefore r = \sqrt{1024} = 4.$$

Let S = ſum of a ſeries of numbers in geometrical progreſſion $= a + ar + ar^2 + ar^3 - - - - - ar^{n-2} + ar^{n-1}$

$$\therefore S - a = ar + ar^2 + ar^3 - - - - - ar^{n-2} + ar^{n-1}$$

and $S - ar^{n-1} = a + ar + ar^2 + ar^3 - - ar^{n-3} + ar^{n-2}$.

Multiply both ſides of the latter equation by r.

$$\therefore rS - ar^n = ar + ar^2 + ar^3 - - - - ar^{n-2} + ar^{n-1}$$

$$\therefore rS - ar^n = S - a$$

$$\therefore rS - S = ar^n - a$$

$$\therefore S \times \overline{r - 1} = a \times \overline{r^n - 1}$$

$$\therefore S = a \times \frac{r^n - 1}{r - 1}.$$

Since $ar + ar^2 + ar^3 - - - - - ar^{n-2} + ar^{n-1}$ is equal to both $S - a$, and $\overline{S - ar^{n-1}} \times r$; the ſum of a ſeries in geometrical progreſſion, wanting the firſt term, is equal to the ſum of all but the laſt term multiplied into the common multiplier.

$$S - A = \overline{S - L} \times r$$

$$\therefore rS - S = rL - A$$

$$\therefore S = \frac{rL - A}{r - 1}.$$

Hence, of the five things S, A, L, n and r, any three being given, the other two may be found: but as the equations, from which they are to be deduced by the common way, would be of a very high order, the two things required are in general beſt diſcovered by logarithms.

$$\text{Since } S = A \times \frac{r^n - 1}{r - 1}$$

$$r^n = \frac{\overline{r - 1}.S + A}{A}$$

$$\therefore n \text{ log. of } r = \text{log. of } \frac{\overline{r - 1}.S + A}{A}$$

$$\therefore n = \text{log. of } \frac{\overline{r - 1}.S + A}{A} \div \text{log. of } r$$

$$\text{Since } S - A = \overline{S - L} \times r$$

$$r = \frac{S - A}{S - L}.$$

TO FIND THE COMMON MULTIPLIER, WHEN THE SUM, FIRST TERM, AND NUMBER OF TERMS ARE GIVEN.

$$S = a + ar + ar^2 + ar^3 \text{ ------- } + ar^{n-1}$$

$$\therefore \frac{S - a}{a} = r + r^2 + r^3 \text{ ------- } + r^{n-1}.$$

Among the dividers of $\frac{S - a}{a}$ will be found one equal to r: and, if r is made equal to $y + 1$, a new equation ariſes, in

in which the known term will be $\frac{S-a}{a} - \overline{n-1}$, or $\frac{S-a-na+a}{a}$ or $\frac{S-na}{a}$. Conſequently, among the dividers of $\frac{S-na}{a}$ will be found one leſs than r by unity. If r is made equal to $y-1$, an equation ariſes, in which the known term will be $\frac{S-a}{a}$ or $\frac{S}{a}$, according as n is an odd or an even number. Conſequently, among the dividers of $\frac{S-a}{a}$ or $\frac{S}{a}$, there will be found one greater than r, by unity. Hence three ſeries of dividers may be made; and in the middle ſeries, one of the dividers, which has a number in each of the other ſeries differing from it by unity, will be equal to r: But in ſome caſes there may be ſeveral dividers in the middle ſeries, anſwering to this poſition; and to ſtrike thoſe out, which will not anſwer the queſtion, the original equation may be referred to.

$$\frac{S-a}{a} = r + r^2 + r^3 + r^4 \text{ - - - - - - - - } + r^{n-1}$$

$$\therefore \frac{S-a}{ar} = 1 + r^1 + r^2 + r^3 \text{ - - - - - - - - } + r^{n-2}.$$

If r is an even number, $1 + r + r^2 + \text{- - -} + r^{n-2}$ is an odd number, and the co-part of r, in the dividers, muſt be an odd number; if r is an odd number, and the number of terms is even, its co-part muſt be an odd number; if the number of terms is odd, its co-part muſt be an even number, conſequently ſeveral dividers may be rejected at the firſt glance. Again, as r increaſes, $\frac{S-a}{ar}$ decreaſes;

decreaſes; and by examining a few terms only of the co-part of r, the higher dividers may be rejected. The proof that the right divider is taken, is eaſy.

$$\frac{S-a}{ar} = 1 + r + r^2 \text{ ---------- } + r^{n-2}$$

$$\therefore \frac{S-a}{ar} - 1 = r \times \overline{1 + r + r^2 \text{ ----- } + r^{n-3}}$$

$$\therefore \overline{\frac{S-a}{ar} - 1 \div r} = 1 + r + r^2 + r^3 \text{ ---- } + r^{n-3}$$

$$\therefore \overline{\frac{S-a}{ar} - 1 \div r - 1} = r \times \overline{1 + r + r^2 \text{ ---- } + r^{n-4}}$$

And thus continually, by taking away 1 from the reſult of each diviſion, the laſt remainder will be equal to r.

Let the firſt term of the ſeries be 21, the ſum 2541, the number of terms 5, what is the common multiplier?

$$\left.\begin{aligned} \frac{S-na}{a} = \frac{2541-105}{21} = \frac{2436}{21} &= 116 \\ \frac{S-a}{a} = \frac{2541-21}{21} = \frac{2520}{21} &= 120 \\ \frac{S-a}{a} &= 120 \end{aligned}\right\} \begin{aligned} &1, 2, 4, \text{ --- } 29 \\ &1, 2, 3, 4, 5, 6 \\ &1, 2, 3, 4, 5, 6. \end{aligned}$$

2 and 3 are the only dividers to be tried, and 2 cannot ſucceed, for in that caſe $\frac{120}{r}$ ſhould be an odd number; therefore the common multiplier is 3, its co-part is 40, which is equal to $1 + 3 + 3^2 + 3^3$.

Let

Let the firſt term of the ſeries be 21, the ſum 7644, the number of terms 6. What is the common multiplier?

$$\frac{S-na}{a} = \frac{7644-126}{21} = \frac{7518}{21} = 358 \text{ - - - } 1, 2 \text{ - - -}$$

$$\frac{S-a}{a} = \frac{7644-21}{21} = \frac{7623}{21} = 363 \text{ - - - } 1, 3 \text{ - - -}$$

$$\frac{S}{a} = \frac{7644}{21} = 364 \text{ - - - } 1, 2, 4 \text{ - -}$$

Three is the common multiplier.

Let the ſum of the ſeries be 159614413, the firſt term 49, and the number of terms 7. What is the common multiplier?

$$\frac{S-na}{a} = \frac{159614413-343}{49} = \frac{159614070}{49} = 3257430\text{-}1,2,3,5,6,10,11,15$$

$$\frac{S-a}{a} = \frac{159614413-49}{49} = \frac{159614364}{49} = 3257436\text{-}1,2,3,4,6,7,12,13$$

$$\frac{S-a}{a} = 3257436\text{-}1,2,3,4,6,7,12,13$$

2, 3, 6, 12 are the dividers to be tried of which 2 and 3 are evidently too ſmall, and 6 is rejected becauſe its co-part ſhould be an odd number; 12 therefore is the required multiplier, and

$$\frac{3257436}{12} = 1 + 12 + 12^2 \text{ - - - - } 12^5$$

$$\frac{3257436}{12} = 271453$$

271453

$$271453 - 1 = 271452 \text{ and } \frac{271452}{12} = 22621$$

$$22621 - 1 = 22620 \text{ and } \frac{22620}{12} = 1885$$

$$1885 - 1 = 1884 \text{ and } \frac{1884}{12} = 157$$

$$157 - 1 = 156 \text{ and } \frac{156}{12} = 13$$

$$\text{And } 13 - 1 = 12.$$

The knowledge of the ſeries both arithmetical and geometrical is neceſſary in calculating the value of money put out to intereſt, of annuities, and of diſcounts.

The intereſt of money is the ſum paid for the uſe of it. The rate of intereſt is the ſum paid for the uſe of a certain ſum for a certain time, as five pounds for the uſe of a hundred pounds for one year.

The intereſt of money is either ſimple or compound. When the intereſt alone is taken at the given rate, it is called ſimple intereſt: but when the intereſt is ſuffered to be joined to the ſum lent, and intereſt is taken upon both ſums, it is called compound intereſt. The ſum originally lent is called the capital or principal; the amount is the whole ſum due at the end of any time of the intereſt and principal together.

Let P = principal, r the intereſt of one pound for one year, n the number of years for which the principal is lent, M the amount. Then $1 : r :: n : nr$ = the intereſt of one pound for n years. Therefore the intereſt of P for n years

years muſt be Pnr, and the amount of principal and intereſt at the end of n years; or $M = P + Pnr = P \times \overline{1 + nr}$.

Hence, of the four numbers P, n, r and M, any three being given, the fourth may be found by a ſimple equation.

$$P = \frac{M}{1 + nr}$$

$$n = \frac{M - P}{Pr}$$

$$r = \frac{M - P}{Pn}.$$

What is the amount of 340*l.* in eight years and a half at 5 per cent. ſimple intereſt?

$$100 : 5 :: 1 : r$$

$$\therefore r = \frac{5}{100} = ,05$$

$$\therefore nr = 8,5 \times ,05 = ,425$$

$$M = P \times \overline{1 + nr} = 340 \times 1,425 = 484,5 = 484l.\ 10s.$$

What was the principal lent, which in 8 years and a half amounted to 484*l.* 10*s.* at five per cent. ſimple intereſt?

$$P = \frac{M}{1 + nr} = \frac{484,5}{1,425} = 340l.$$

For

For three hundred and forty pounds lent at the rate of five per cent. ſimple intereſt, 484*l.* 10*s.* was paid. How many years had it been lent?

$$n = \frac{M - P}{Pr} = \frac{484{,}5 - 340}{340 \times {,}05} = \frac{144{,}5}{17} = 8{,}5 = 8\tfrac{1}{2} \text{ years.}$$

Three hundred and forty pounds in eight years and a half amounted to 484l. 10*s.* at ſimple intereſt. What was the rate of intereſt?

$$r = \frac{M - P}{Pn} = \frac{484{,}5 - 340}{340 \times 8{,}5} = \frac{144{,}5}{20 \times 17 \times 8{,}5} = \frac{8{,}5}{20 \times 8{,}5} = \frac{1}{20} = {,}05.$$

What is the amount of an annuity or penſion left unpaid any number of years, allowing ſimple intereſt upon each ſum from the time it becomes due?

Let b = the annuity, whoſe intereſt at the end of the ſecond year is br, and the amount of the intereſt at the end of $n - 1$ year is $\overline{n-1} \cdot br$. At the end of the ſecond year b is again due, and the amount of the intereſt upon this ſum, at the end of $n - 2$ years, is $\overline{n-2} \cdot br$. In the ſame manner the amount of the intereſt upon the third ſum due is $\overline{n-3} \cdot br$, and the amount of the intereſt on all the ſums due is $br \times \overline{n-1+n-2+n-3 \cdots +1}$, or $br \times n \cdot \frac{n-1}{2}$. The ſum of the annuities to be paid is nb. Therefore the amount of the annuities, with their reſpective intereſts, is $nb + br \times n \cdot \frac{n-1}{2}$.

What is the preſent value of an annuity, to continue for a certain number of years unpaid, allowing ſimple intereſt for the money each year when it becomes due?

Let P be the preſent value; then if P, with its intereſt at the end of the years given, amounts to the ſame ſum that the annuity with its intereſt does, they are of equal value. The amount of P in n years is $P \times \overline{1 + nr}$; and the amount of the annuity in the ſame time is by the laſt propoſition $nb + br \times n \times \frac{n-1}{2}$.

$$\therefore P \times \overline{1 + nr} = nb + br \times n . \frac{n-1}{2}$$

$$\therefore P = \frac{nb + br \times n . \frac{n-1}{2}}{1 + nr.}$$

What is the preſent value of an annuity of one pound a year, with the ſimple intereſt on each pound as it became due, to be paid at the end of twenty years, the intereſt being at the rate of five per cent?

$$P = \frac{nb + br \times n . \frac{n-1}{2}}{1 + nr}$$

$$P = \frac{20 + ,05 \times \frac{20 \times 19}{2}}{1 + 20 \times ,05} = \frac{20 + ,05 \times 190}{2} =$$

$$\frac{29,50}{2} = 14,75 = 14l.\ 15s.$$

Suppoſe

Suppoſe twenty pounds to have been paid down for an annuity of one pound a year, and the intereſt to be five per cent. after, How many years is the purchaſer to receive the amount of the annuity and its intereſt, and what ſum will he receive?

$$P = \frac{nb + br \times n \cdot \frac{n-1}{2}}{1 + nr}$$

$$\therefore 20 = \frac{n + ,05 \times n \cdot \frac{n-1}{2}}{1 + n \times ,05}$$

$$\therefore 20 + n = n + \frac{n^2 \times ,05}{2} - \frac{n \times ,05}{2}$$

$$\therefore \frac{40}{,05} = n^2 - n$$

$$\therefore n^2 - n = 800$$

$$\therefore n = \sqrt{800 + \tfrac{1}{4}} + \tfrac{1}{2} = 28,784 \text{ nearly.}$$

The ſum which the purchaſer is to receive is the annuity for 28,784 years, or 28*l.* 15*s.* 8*d.* and the intereſt of each pound as it became due, or 19*l.* 19*s.* 10*d.* The amount therefore to be received is 48*l.* 15*s.* 6*d.* The intereſt of 20*l.* for 28,784 years is 28*l.* 15*s* 8*d.* and the ſum advanced was 20*l.* The amount therefore of the money he was out of pocket is 48*l.* 15*s.* 8*d.* The difference between the two amounts is two pence, owing to the number *n* not having been more accurately aſcertained.

By

By the ſame equation $P = \frac{nb + br \times n \cdot \frac{n-1}{2}}{1 + nr}$ the number of years may be calculated, at the end of which a perſon who has given a thouſand pounds for an annuity of a farthing a year is to receive the amount of the annuity and its intereſt, and the number of years may be made ſo great, that an annuity of a farthing a year, with its intereſt, ſhall exceed any given ſum whatever.

To calculate the amount of a ſum of money lent at compound intereſt, let a = the amount of one pound at the end of one year. Therefore, ſince one pound with its intereſt at the end of a year is equal to a, at the end of the 2d year its amount is equal to a^2, at the end of the third to a^3, at the end of the nth year to a^n. For

$$1 : a :: a : a^2 :: a^2 : a^3 \text{ - - - - - - } a^{n-1} : a^n$$

Hence the amount of P pounds at the end of n years is $Pa^n = M$; and of the four things P, a, n, M, any three being given, the fourth may be found.

$$P = \frac{M}{a^n}$$

$$a = \sqrt[n]{\frac{M}{P}}$$

$$n = \frac{\log. M - \log. P}{\log. a.}$$

When n is a high number, all the equations will be ſolved with greater eaſe by means of logarithms.

$$\text{Log. P} = \text{log. M} - n \text{ log. } a$$

$$\text{Log. M} = \text{log. P} + n \text{ log. } a$$

$$\text{Log. } a = \frac{\text{log. M} - \text{log. P}}{n}.$$

To calculate the value of an annuity at compound intereſt, let P be the preſent value, b the annuity, a the amount of one pound at the end of one year, and n the number of years in which the annuity is to be paid.

The amount of the firſt payment of the annuity at the end of the n years is ba^{n-1}, of the ſecond payment is ba^{n-2}, of the third ba^{n-3}, and ſo on, and of the laſt is b. Hence the amount of all the payments is

$$b \times \overline{1 + a + a^2 + a^3 - - - - - - + a^{n-1}}, \text{ or } b \times \frac{a^n - 1}{a - 1}.$$

Pa^n is the amount of P at the end of n years.

$$\therefore Pa^n = b \times \frac{a^n - 1}{a - 1}$$

$$\therefore P = b \times \frac{1 - \frac{1}{a^n}}{a - 1}.$$

If the number of years is very great, $P = b \times \frac{1}{a - 1}$ without material error.

Hence of the four things P, a, n and b, any three things being given, the fourth may be found.

Inſtead

Inſtead of being multiplied by a given number, numbers in geometrical progreſſion may have a common divider. Let A B C D E - - - - I K L be a ſeries of this kind, and let r be the common divider. Then, if the ſeries is in a progreſſion with each term decreaſing, $B = \frac{A}{r}, C = \frac{B}{r} = \frac{A}{r^2}, D = \frac{C}{r} = \frac{A}{r^3}$ - - - - - - - and the laſt term $L = \frac{A}{r^{n-1}}$. A ſeries therefore of this kind may be thus written down:

$$A \times 1 + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} \text{ - - - - - - - } \frac{1}{r^{n-2}} + \frac{1}{r^{n-1}}.$$

The product of any two terms, equally diſtant from the firſt and laſt terms, is equal to the product of the firſt and laſt terms.

$$B = \frac{A}{r} \qquad K = \frac{A}{r^{n-2}}$$

$$\therefore B \times K = \frac{A}{r} \times \frac{A}{r^{n-2}} = A \times \frac{A}{r^{n-1}} = A \times L$$

$$C = \frac{A}{r^2} \qquad I = \frac{A}{r^{n-3}}$$

$$\therefore C \times I = \frac{A}{r^2} \times \frac{A}{r^{n-3}} = A \times \frac{A}{r^{n-1}} = A \times L.$$

If the number of terms is odd, the middle term is equal to the ſecond root of the product of the firſt and laſt terms.

$$M = \frac{A}{r^{\frac{n-1}{2}}}$$

$$\therefore M^2 = \frac{A^2}{r^{n-1}} = A \times \frac{A}{r^{n-1}} = A \times L$$

$$\therefore M = \sqrt[2]{A \times L}.$$

From the equation $L = \frac{A}{r^{n-1}}$, if three out of the four things L, A, r and n are given, the fourth may be found.

Let S be the ſum of the ſeries.

$$\text{Then } S = A + \frac{A}{r} + \frac{A}{r^2} + \frac{A}{r^3} \text{ ------ } + \frac{A}{r^{n-1}}$$

$$\therefore S - A = \frac{A}{r} + \frac{A}{r^2} + \frac{A}{r^3} \text{ ------ } + \frac{A}{r^{n-1}}$$

$$\overline{S - L} \times \frac{1}{r} = \frac{A}{r} + \frac{A}{r^2} + \frac{A}{r^3} \text{ ------ } \frac{A}{r^{n-1}}$$

$$\therefore S - A = \overline{S - L} \times \frac{1}{r}$$

$$\therefore rS - rA = S - L$$

$$\therefore rS - S = rA - L$$

$$\therefore S = \frac{rA - L}{r - 1} = \frac{rA - \frac{A}{r^{n-1}}}{r - 1} = rA \times \frac{1 - \frac{1}{r^n}}{r - 1}.$$

Hence the ſum of any number of terms of a ſeries of this kind cannot be greater than $\frac{rA}{r-1}$, and the greater the number of terms is, the nearer does the ſum approach to $\frac{rA}{r-1}$. For the difference between $\frac{rA}{r-1}$ and the real ſum of

of the ſeries is $\frac{A}{r^{n-1}} \times \frac{1}{r-1}$, and this latter number may be made ſmaller than any number aſſigned, by increaſing the number of terms. The number $\frac{r \cdot A}{r-1}$, is called the limit of the ſeries.

What is the limit of the ſeries $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ - - - - - - - - - - - - ?

In this caſe $A = 1$ and $r = 2$, therefore the limit is $\frac{2 \times 1}{2 - 1} = 2$.

What is the difference between the ſum of five terms of the above ſeries and its limit?

The difference is $\frac{1}{2^{n-1}} \times \frac{1}{2-1} = \frac{1}{2^4} = \frac{1}{16}$.

This is ſeen alſo from finding the ſum of five terms.

$$S = 2 \times 1 \times \overline{1 - \frac{1}{2^5}} = 2 \times \overline{1 - \frac{1}{32}} = \frac{2 \times 31}{32} = \frac{31}{16} = 1\frac{15}{16}.$$
$$\overline{2 - 1}$$

The difference between $1\frac{15}{16}$ and the limit 2 is $\frac{1}{16}$.

What is the limit of the ſeries $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \text{- - - - - - - ?}$

The limit is $\frac{3 \times 1}{3 - 1} = \frac{3}{2} = 1\frac{1}{2}$.

What is the limit of the ſeries $\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \frac{1}{10000} + \text{-------}$?

$$\text{The limit is } \frac{\frac{1}{10} \times 10}{9} = \frac{1}{9}.$$

Various laws may be laid down for the progreſſive terms of a ſeries; and to find the ſum of any number of terms, various methods have been invented. To enumerate them all would carry us beyond the bounds of the preſent undertaking; and it will be ſufficient, by a few inſtances, to give a general idea of methods, and to guard the learner againſt ſome miſtakes, into which, from many books on this ſubject, he is in danger of falling.

Let it be required to find the ſum of any number of terms in the ſeries $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} \text{-------} + \frac{1}{n.n+1}$. This ſeries may be derived from ſubtracting one ſeries from another. Thus,

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \text{---} + \frac{1}{n-1} + \frac{1}{n}$$

$$\therefore S - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \text{---} + \frac{1}{n}.$$

Subtract the lower ſeries from the upper.

$$\therefore S - S + 1 = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} \text{---} + \frac{1}{n-1.n} + \frac{1}{n}$$

$$\therefore 1 - \frac{1}{n} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} \text{---} + \frac{1}{n-1.n}.$$

Let m = number of terms in this laſt ſeries. Then $m = n - 1$. $\therefore n = m + 1$.

$$\therefore 1 - \frac{1}{m+1} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} \text{---} + \frac{1}{m.m+1}.$$

Since

Since $\frac{1}{m+1}$ may be made leſs than any aſſignable number, the limit of this ſeries is 1.

Let it be required to find the ſum of any number of terms of the ſeries $\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}+\frac{1}{4\cdot6}+\&c.+\frac{1}{n.\overline{n+2}}$.

$$S = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\text{ ---------- }+\frac{1}{n-1}+\frac{1}{n}$$

$$\therefore S-1-\frac{1}{2}=\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\text{ ----- }+\frac{1}{n}$$

$$\therefore 1+\frac{1}{2}=\frac{2}{1\cdot3}+\frac{2}{2\cdot4}+\frac{2}{3\cdot5}\text{ ---- }+\frac{2}{n.\overline{n-2}}+\frac{1}{n-1}+\frac{1}{n}$$

$$\therefore \frac{3}{2}-\frac{1}{n-1}-\frac{1}{n}=\frac{2}{1\cdot3}+\frac{2}{2\cdot4}+\frac{2}{3\cdot5}\text{ --- }+\frac{2}{n.\overline{n-2}}$$

$$\therefore \frac{3}{4}-\frac{2n-1}{2n.\overline{n-1}}=\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}\text{ ------- }+\frac{1}{n.\overline{n-2}}.$$

Let m = the number of terms in this lower ſeries, and as there are more terms by two in the firſt than the laſt ſeries, $m = n-2$, and $\frac{3}{4}-\frac{2m+3}{2.\overline{m+2}.\overline{m+1}}=\frac{1}{1\cdot3}+\frac{1}{2\cdot4}+\frac{1}{3\cdot5}\text{ ----- }\frac{1}{m.\overline{m+2}}$.

The limit of this ſeries is $\frac{3}{4}$.

In general the ſum of any number of terms of a ſeries under this form $\frac{1}{a.\overline{a+b}} + \frac{1}{\overline{a+b}.\overline{a+2b}}$ $+ \frac{1}{\overline{a+2b}.\overline{a+3b}}$ ----- $+ \frac{1}{\overline{a+n-1b}.\overline{a+n}.b}$ may be found by a ſimilar proceſs.

$$S = \frac{1}{a} + \frac{1}{a+b} + \frac{1}{a+2b} + \text{-----} + \frac{1}{a+n-1.b}$$

$$\therefore S - \frac{1}{a} = \frac{1}{a+b} + \frac{1}{a+2b} + \frac{1}{a+3b} \text{---} + \frac{1}{a+n-1.b}$$

$$\therefore \frac{1}{a} = \frac{b}{a.a+b} + \frac{b}{a+b.a+2b} + \text{----} \frac{b}{a+n-2.b.a+n-1b}$$

$$+ \frac{1}{a+n-1.b}.$$

$$\frac{1}{ab} - \frac{1}{b.a+n-1.b} = \frac{1}{a.a+b} + \frac{1}{a+b.a+2b} \text{---} \frac{1}{a+n-2.b.a+n-1.b}.$$

Let m = the number of terms in the laſt ſeries.

$$\therefore n = m + 1.$$

$$\frac{1}{b} \times \overline{\frac{1}{a} - \frac{1}{a+mb}} = \frac{1}{a.a+b} + \text{---} \frac{1}{a+m-1.b.a+mb} =$$

$$\frac{mb}{ab.\overline{a+mb}} = \frac{m}{a.\overline{a+mb}}.$$

The greater the number of terms, the nearer will $\frac{m}{a+mb}$ approach to $\frac{1}{b}$; therefore the limit of a ſeries of this form is $\frac{1}{ab}$.

ON

ON THE APPLICATION OF ALGEBRA TO GEOMETRY.

SINCE algebra is the ſcience of numbers, it can be applied to thoſe parts only of other arts and ſciences in which numbers are concerned; and it cannot produce any concluſion but what relates to numbers. If it is applied to geometry, the poſition of lines, ſurfaces and ſolids cannot be conſidered: but, if a queſtion involves the numbering of lines, ſurfaces and ſolids or of their parts, the queſtion, inaſmuch as the numbers merely are concerned, may be a queſtion of algebra. Thus, let a point moving up and down the line AB, fig. 1, at the ſame time that the line is carried along another line CD, deſcribe the curve AKEFGH; and let the two motions be ſuch, that the line KI, drawn perpendicularly to the axis CD, ſhall always bear ſome relation to CI, which may be determined by an equation. Thus, let KI, called y, be meaſured by feet, and CI, called x, be meaſured in feet alſo: and let a, b, c, d, &c. repreſent known numbers. If $y^m = ax^n \pm bx^{n-1} \mp cx^{n-2} \pm dx^{n-3}$ - - - &c, it is the part of algebra to find out the value of y in terms of x and the known quantities, and to find out x in terms of y and the known quantities. In other words, it is the part of algebra to ſolve the equation, y or x being given. But algebra has nothing to do with the poſition of KI, whether it is above or below the line CD; whether it is to the right or left of the point C; whether it makes a right or oblique angle to any line, or, in ſhort, whether it meets any other line or not.

When the relation between CI and KI can be determined

mined by an equation, the line deſcribed by the point may be called an algebraical line or curve. If the equation is a ſimple equation, the line is of the firſt order: if it is an equation of the ſecond order, the line is ſaid to be of the ſecond order: if it is an equation of the nth order, the line is ſaid to be of the nth order. CI is in general called the abſciſs, or ſegment of the axis; IK is called the ordinate. The ordinate may make an oblique angle with the axis, and when the angle is determined and at the point I a line is drawn to an indefinite length, making the given angle with the axis CI; the part IK, to be cut off, may be determined by algebra, if the equation is given.

Let a point in the line HI, fig. 2, moving upon the line CD, and preſerving the given angle HID, deſcribe another line, and let the equation between CI and IK be of this form, $ay = bx - c$; then, for each poſition of I and K, the lines IK and CI may be determined.

Firſt, let IK = o.

$$\therefore bx - c = o$$

$$\therefore bx = c$$

$$\therefore x = \frac{c}{b}.$$

Let CL $= \frac{c}{b}$ feet. Therefore when I is at L, K, L and I muſt coincide. As I moves from L towards D, CI is conſtantly increaſing; therefore IK muſt conſtantly increaſe: for x increaſing, $bx - c$ increaſes; that is, ay increaſes; that is, y increaſes. If IH is between L and C, IK will have no number correſponding to it: that is, there

can

can be no ſuch line as IK upon the things given in the equation. For in this caſe $ay = bx - c$; but x is now leſs than $\frac{c}{b}$. $\therefore bx > c$. Therefore c cannot be taken away from bx, and conſequently ay has no number correſponding to it. Hence the line deſcribed by the point K begins at L, and the diſtance between any point in the line LK and the line LD, meaſured in the direction KI, conſtantly increaſes.

$$LI = CI - CL$$

$$\therefore LI = x - \frac{c}{b} = \frac{bx - c}{b}$$

$$ay = bx - c$$

$$\therefore y = \frac{bx - c}{a} = IK$$

$$\therefore IK : LI :: \frac{bx - c}{a} : \frac{bx - c}{b} :: b : a.$$

Therefore IK is to LI in a conſtant ratio, and conſequently the line deſcribed by the point K muſt be a ſtraight line.

Let the line deſcribed by the motion of a point in a line perpendicular to the axis be ſuch, that $y^2 = ax$. Hence if $y = 0$. $x = 0$. As x increaſes, y increaſes; and there is no limit to their increaſe.

$y = \sqrt{ax}$, and when $x = a$, $y = \sqrt{a^2} = a$.

Suppoſe CE, fig. 3, to be equal to a, then EH = CE.

Thi

This is a line of the ſecond order, and it is a ſemi-parabola.

Let the equation be $y^2 = \frac{c^2}{d} \times \overline{2ax - x^2}$. If $x = 0$, $y = 0$. And if $x = 2a$, $2ax - x^2 = 0$. $\therefore y = 0$. And conſequently in two ſituations of I, IK will be equal to nothing. As x increaſes, $2ax - x^2$ muſt firſt increaſe and then decreaſe, and it will be greateſt when $x = a$: conſequently y firſt increaſes, and then decreaſes; and is greateſt when $x = a$.

The line will therefore be of the kind ſeen in fig. 4.

$$CD = 2a, \quad CM = a.$$

At C and D, IK vaniſhes, at M it is greateſt; in moving from C to M the line IK conſtantly increaſes; in moving from M to D it conſtantly decreaſes. The line is of the ſecond order, and is a ſemi-ellipſe.

$$\text{Let } y^2 = a^2 - x^2,$$

$$\text{If } x = 0, \; y^2 = a^2. \quad \therefore y = a.$$

$$\text{If } y = 0, \; a^2 - x^2 = 0. \quad \therefore x = a.$$

x cannot be greater than a, for then x^2 could not be taken from it; conſequently there will be no IK, if x is made greater than a.

Let IK, fig. 5, move perpendicularly upon the line CD from C to D, and let $CD = a$. When I is at C, $x = 0$. $\therefore$ IK becomes $CH = CD$. As IK moves from C to D, it conſtantly decreaſes; and when I is at D, $y = 0$. Therefore the line deſcribed

deſcribed by K muſt lie within H and D, and is of the ſecond order. It is the quadrant of a circle.

Let the equation be $x^4 - a^2x^2 + a^2y^2 = 0$.

$$\therefore a^2x^2 - x^4 = a^2y^2.$$

If $x = 0$, $y = 0$; and if $y = 0$, x may be nothing; or ſince $a^2x^2 - x^4 = x^2 \times \overline{a^2 - x^2}$; when $x = a$, $a^2 - x^2 = 0$. Therefore IK vaniſhes in two ſituations, firſt when I is at C and when CI $= a$. x cannot be greater than a, becauſe then $a^2 - x^2$ would not be a number. As x increaſes from nothing, $a^2 - x^2$ decreaſes: therefore $x^2 \times \overline{a^2 - x^2}$ firſt increaſes and then decreaſes.

Let CD, fig. 6, $= a$, and the curve may be formed upon the line CD, by finding KI for every point I from the equation $y = \frac{x\sqrt{a^2 - x^2}}{a}$.

Let $ax - x^2 = y^2$.

$y = 0$, when $x = 0$ or a.

An equation of the ſecond order in this form may admit of two values for the unknown number x: but there cannot on this account be two lengths of IK; for either $\frac{a}{2} - x$, or $x - \frac{a}{2}$ may be equal to $\sqrt{\frac{a^2}{4} - y^2}$: and when CI is leſs than $\frac{a}{2}$, then only $\frac{a}{2} - x$ can be equal to $\sqrt{\frac{a^2}{4} - y^2}$; and when CI is greater than $\frac{a}{2}$, only $x - \frac{a}{2}$ can be equal to $\sqrt{\frac{a^2}{4} - y^2}$.

From

From the equation to a curve it may be delineated, and alſo many of its properties may be diſcovered. Thus it may be proved algebraically, that the diſtance of a point in the parabola from the focus is equal to its perpendicular diſtance from the directrix.

The equation is $y^2 = ax$.

Let AS, fig. 7, $= b$, then $y^2 = 4bx$

and $SE = x - b$

$$\therefore SE^2 = \overline{x - b}^2$$

But $SP^2 = SE^2 + EP^2$

$$\therefore SP^2 = \overline{x - b}^2 + y^2 = x^2 - 2bx + b^2 + 4bx$$

$$\therefore SP^2 = x^2 + 2bx + b^2$$

$$\therefore SP = x + b = AE + AS = EH = PM.$$

In the ſame manner the chief properties of the conic ſections and the curves of higher orders have been inveſtigated by various writers, whoſe works may be read with profit, if the learner keeps in mind only what has been before advanced, that an algebraical ſolution can refer only to number, and that the poſition of the lines with reſpect to each other does not lie within the province of algebra.

In geometry, ſurfaces are compared with each other by means of rectangles, and the number of ſquare feet in a rectangle depends on the length of the ſides containing the right angle. Theſe may be determined in numbers by meaſuring the length of each ſide, and multiplying the numbers together; for the number of ſquare feet in the

rectangle will be equal to the product of the multiplication of the number of feet in one ſide into the number of feet in the other. Let a = number of feet in one ſide, and b = number of feet in the other, then ab = the number of ſquare feet in the rectangle. In the ſame manner abc = number of cubical feet in a ſolid equal to a cube whoſe ſides are a, b, and c feet in length.

Hence, if the number of ſquare feet in a ſurface, or of ſolid feet in a cube, is given, and the number of feet in the lines forming theſe ſurfaces and ſolids is repreſented by algebraical terms, the lines compoſing theſe ſurfaces and ſolids may be frequently found. If the number of ſquare feet in a rectangle is equal to a, then $b \times c = a$; and if c one ſide is given, then the other ſide b will be found by dividing a by c. In the ſame manner, if the number of cubical feet $= a$, and two of the ſides of the cube are given, the third is found by dividing a by $b \times c$.

If the ſide of a ſquare is required to be found when the ſquare itſelf is equal to a, make $x \times x$, or $x^2 = a$. $\therefore x = \sqrt{a}$. Find the ſecond root of a, and the reſult is the number of feet in the line x.

Let a cube whoſe ſide is required $= a$, then $x \times x \times x$ or $x^3 = a$. $\therefore x = \sqrt[3]{a}$. Find the third root of a, and the reſult is the number of feet in the line x.

Hence in geometrical problems where ſquares and cubes are the ſubjects, the problem may be oftentimes reſolved algebraically, by conſidering the number of ſquare and ſolid

ſolid feet in the geometrical figure, and making that the baſis of the algebraical problem.

Let it be required to divide a line into two parts, ſuch that the rectangle under the two parts ſhall be equal to the ſquare on another line.

Let AB, GD, fig. 8, be the given lines, of which AB is to be divided in ſuch a manner, that the rectangle under AE and BE ſhall be equal to the ſquare on GD.

$$\left.\begin{array}{l} \text{Let AB} = a \\ \text{BE} = x \\ \text{GD} = b \end{array}\right\} \text{feet.}$$

$$\text{Then } x \times \overline{a - x} = b^2$$

$$\therefore ax - x^2 = b^2$$

$$\therefore \frac{a^2}{4} - ax + x^2 = \frac{a^2}{4} - b^2$$

$$\therefore \sqrt{\frac{a^2}{4} - ax + x^2} = \sqrt{\frac{a^2}{4} - b^2}$$

$$\therefore \frac{a}{2} - x \text{ or } x - \frac{a}{2} = \sqrt{\frac{a^2}{4} - b^2}$$

$$\therefore x = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b^2} \text{ or } \frac{a}{2} - \sqrt{\frac{a^2}{4} - b^2}.$$

$x =$

e = BE, or Be ſegments of the diameter of a circle, ef and EF being equal to GD; for then C being the centre, CE will be equal to $\sqrt{\frac{a^2}{4} - b^2}$, and BE, B$e$ = BC ± CE.

If a ſtraight line is divided into two parts, the ſquare of the whole line is equal to the ſquares of the two parts together with twice the rectangle under the two parts.

Let a and b be the number of feet in the two parts. Then $a + b$ is the number of feet in the whole line. $\therefore \overline{a + b}^2$ is the number of ſquare feet in the whole line.

$$\text{But } \overline{a + b}^2 = a^2 + 2ab + b^2.$$

a^2 is the number of ſquare feet in a ſquare made upon a line a feet in length.

b^2 is the number of ſquare feet in a ſquare made upon a line b feet in length.

And $2ab$ is the number of feet in two rectangles made with lines reſpectively a and b feet in length.

The learner may find ſeveral ſimilar inſtances in the ſecond book of Euclid, which will afford him uſeful practice; and from an inveſtigation of the mode purſued with them, he will find it very eaſy to underſtand the writers who have applied algebra to deeper geometrical problems.

AN

APPENDIX,

BY

FRANCIS MASERES, Esq. F.R.S.

CURSITOR BARON OF HIS MAJESTY'S COURT OF EXCHEQUER.

Algebram verò, ut solet doceri, *animadverti certis regulis et numerandi formulis ita esse contentam, ut videatur potiùs Ars quædam confusa, cujus usu Ingenium quodam modo turbatur et obscuratur, quàm Scientia, quâ excolatur et perspicacius reddatur.*

Des Cartes in his Dissertation *De Methodo*, page 11.

OF THE SOLUTION OF CERTAIN CUBICK EQUATIONS, OR EQUATIONS OF THE THIRD ORDER, BY CARDAN'S RULES.

Article I. TWO rules were publiſhed by *Cardan*, a learned phyſician and mathematician at *Milan* in *Italy*, in the year 1545, for reſolving cubick equations of the two following forms, to wit, $x^3 + bx = c$ and $x^3 - bx = c$. The former of theſe rules, which relates to the equation $x^3 + bx = c$, is always ſufficient to reſolve it, whatever be the magnitudes of b, the co-efficient of x, and of c, the abſolute term of the equation: but the ſecond rule, which relates to the equation $x^3 - bx = c$, will reſolve it only when the abſolute term c is greater than $\frac{2b \times \sqrt{b}}{3\sqrt{3}}$, or cc is greater than $\frac{4b^3}{27}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$. Neither of theſe rules was invented by Cardan; but the former of them had been invented about 30 years before Cardan publiſhed it, that is, about the year 1515, by *Scipio Ferreus*, of *Bononia*, or *Bologna*, in *Italy*, as Cardan himſelf informs us; and the latter rule had been lately invented and communicated to Cardan by *Nicholas Tartalea*, or *Tartaglia*, a learned Italian, who was Profeſſor of Mathematicks at *Venice*. See upon this ſubject *Montucla's Hiſtoire des Mathématiques*, vol. i. pages 462, and 479, 480, and 481, and Dr. Hutton's Mathematical Dictionary.

CARDAN'S FIRST RULE.

Art. II. THE former of theſe rules, or that which relates to the equation $x^3 + bx = c$, is as follows; to wit, x is $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} \; \frac{-b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}}$; or, if, for the ſake of abridging this complicated expreſſion, we ſubſtitute the letter e inſtead of $\frac{c}{2}$, and ss inſtead of $\frac{b^3}{27} + \frac{cc}{4}$, and conſequently s inſtead of $\sqrt{\frac{b^3}{27} + \frac{cc}{4}}$, we ſhall have $x = \sqrt[3]{e + s} - \frac{b}{3\sqrt[3]{e + s}}$, or $x = \sqrt[3]{s + e} - \frac{b}{3\sqrt[3]{s + e}}$.

And there are likewiſe two other expreſſions for the value of x in this equation $x^3 + bx = c$, which may be derived from the ſame inveſtigation, by means of which we obtain the foregoing expreſſion $\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} \; \frac{-b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}}$, or $\sqrt[3]{e + s} - \frac{b}{3\sqrt[3]{e + s}}$, or $\sqrt[3]{s + e} - \frac{b}{3\sqrt[3]{s + e}}$. Theſe expreſſions are $\frac{b}{3\sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$,

or

or $\frac{b}{3\sqrt[3]{-e+s}} - \sqrt[3]{-e+s}$, or $\frac{b}{3\sqrt[3]{s-e}} - \sqrt[3]{s-e}$, and $\sqrt[3]{\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}} - \sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}}$, or $\sqrt[3]{e+s} - \sqrt[3]{-e+s}$, or $\sqrt[3]{s+e} - \sqrt[3]{s-e}$. All these three different expressions will be equal to x, or the root of the equation $x^3 + bx = c$, as will easily appear from the investigation of them, which will be given in the following articles.

CARDAN'S SECOND RULE.

Art. III. The latter rule, or that which relates to the equation $x^3 - bx = c$, and which was invented by *Nicholas Tartalea*, is as follows. If c be greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or cc be greater than $\frac{4b^3}{27}$, or $\frac{cc}{4}$ be greater than $\frac{b^3}{27}$, x, or the root of the equation $x^3 - bx = c$, will be equal to either of the three following expressions; to wit,

1st, $\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}-\frac{b^3}{27}}} + \frac{b}{3\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}}$,

or 2dly, $\frac{b}{3\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}} + \sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}$,

or, 3dly, $\sqrt[3]{\frac{c}{2}+\sqrt{\frac{cc}{4}-\frac{b^3}{27}}} + \sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}$;

or, if, for the ſake of brevity, we ſubſtitute e inſtead of $\frac{c}{2}$ and ss inſtead of $\frac{cc}{4} - \frac{b^3}{27}$, and conſequently s inſtead of $\sqrt{\frac{cc}{4} - \frac{b^3}{27}}$ in the ſaid expreſſions, we ſhall have x equal to either of the three following expreſſions, to wit, 1ſt, $\sqrt[3]{e+s} + \frac{b}{3\sqrt[3]{e+s}}$, and 2dly, $\frac{b}{3\sqrt[3]{e-s}} + \sqrt[3]{e-s}$, and 3dly, $\sqrt[3]{e+s} + \sqrt[3]{e-s}$. We will now proceed to give the inveſtigations of theſe two celebrated rules.

THE INVESTIGATION OF CARDAN'S FIRST RULE, WHICH RELATES TO THE RESOLUTION OF THE EQUATION $x^3 + bx = c$.

Art. IV. In order to inveſtigate the firſt rule of Cardan, which relates to the equation $x^3 + bx = c$, we muſt premiſe the following very evident principle; to wit, that, if there be taken any two numbers whatſoever, or any line and rectangle whatſoever, denoted by the letters k and l, it is poſſible for two other numbers, or two lines, to exiſt, (which we will call g and h, whereof let g be the greater,) which are of ſuch magnitudes that their difference $g - h$ ſhall be equal to k, and their product, or rectangle, gh, ſhall be equal to l.

For let z and v be two variable lines, or numbers, of which z is the greater, and of which the difference $z - v$ is

is equal to the given quantity k. It is evident that v, the leſſer of theſe quantities, may be as ſmall, or as nearly equal to nothing, as we pleaſe, and yet may differ from the greater quantity z by the given quantity k; and it is evident likewiſe that it may be as great as we pleaſe, and yet may differ from the greater quantity z by the ſame difference k. Therefore the rectangle, or product, zv, (the magnitude of which evidently depends upon the magnitude of v as well as of z) may be as ſmall or as great as we pleaſe, and yet the quantity z and v may have the ſame given difference k. Therefore zv may be equal to the given quantity l at the ſame time that $z - v$ is equal to k; or z and v may be taken of ſuch magnitudes (which we will call g and h,) that their difference $z - v$, or $g - h$, ſhall be equal to k, and their rectangle, or product, zv, or gh, ſhall be equal to l.

Q. E. D.

This being premiſed, the firſt rule of Cardan above-deſcribed may be inveſtigated, or the problem of reſolving the equation $x^3 + bx = c$ may be ſolved, in the following manner.

PROBLEM I.

Art. V. To reſolve the equation $x^3 + bx = c$.

SOLUTION.

Let g and h be two quantities of ſuch magnitudes that their difference $g - h$ ſhall be equal to x, and their rectangle,

angle, or product, gh, ſhall be equal to $\frac{b}{3}$, it having been proved in the laſt article that it is poſſible for two ſuch quantities to exiſt. Then it is evident that we ſhall have $3gh = b$, and $x^3 = g - h)^3 = g^3 - 3g^2h + 3gh^2 - h^3 = g^3 - 3gh \times \overline{g-h} - h^3 = g^3 - h^3 - 3gh \times \overline{g-h}$, and $bx = b \times \overline{g-h}$, and conſequently $x^3 + bx = g^3 - h^3 - 3gh \times \overline{g-h} + b \times \overline{g-h}$, or (becauſe $3gh$ is $= b$, and conſequently $3gh \times \overline{g-h}$ is $= b \times \overline{g-h}$) $x^3 + bx = g^3 - h^3 - b \times \overline{g-h} + b \times \overline{g-h} = g^3 - h^3$. But $x^3 + bx$ is $= c$. Therefore $g^3 - h^3$ will alſo be $= c$.

But, becauſe $3gh$ is $= b$, it follows that h will be $= \frac{b}{3g}$, and conſequently that h^3 will be $= \frac{b^3}{27g^3}$.

Therefore $g^3 - h^3$ will be $= g^3 - \frac{b^3}{27g^3}$.

But it has been ſeen that $g^3 - h^3$ is $= c$.

Therefore $g^3 - \frac{b^3}{27g^3}$ will alſo be $= c$.

Therefore (multiplying all the terms into g^3,) we ſhall have $g^6 - \frac{b^3}{27} = cg^3$, and (adding $\frac{b^2}{27}$ to both ſides,) $g^6 = cg^3 + \frac{b^3}{27}$, and (ſubtracting cg^3 from both ſides,) $g^6 - cg^3 = \frac{b^3}{27}$; which equation, though it riſes to the 6th power of the unknown quantity g, is of a ſimilar form to a quadratick equation, and is therefore capable of being reſolved in the ſame manner as a quadratick ſo far as to obtain the value

value of g^3, or the cube of the unknown quantity g. We will therefore add $\frac{cc}{4}$, or the ſquare of $\frac{c}{2}$, or of half the co-efficient of g^3, to both ſides of this equation; and we ſhall then have $g^6 - cg^3 + \frac{cc}{4} = \frac{b^3}{27} + \frac{cc}{4}$. Therefore the ſquare-root of $g^6 - cg^3 + \frac{cc}{4}$ will be equal to the ſquare-root of $\frac{b^3}{27} + \frac{cc}{4}$, that is, $g^3 - \frac{c}{2}$ will be $= \sqrt{\frac{b^3}{27} + \frac{cc}{4}}$.

Therefore, if we add $\frac{c}{2}$ to both ſides, we ſhall have $g^3 = \frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}$; and conſequently g will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$.

But we have ſeen that b is $= \frac{b}{3g}$.

Therefore b will be $= \frac{b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}}$, and conſequently $g - b$ will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} \frac{-b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}}$, that is, x, or the root of the propoſed equation $x^3 + bx = c$, will be $=$

$$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} - \frac{b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}}.$$

Q. E. I.

If e be put $= \frac{c}{2}$, and s be put $= \sqrt{\frac{b^3}{27} + \frac{cc}{4}}$, this expreſſion will be $= \sqrt[3]{e + s} - \frac{b}{3\sqrt[3]{e + s}}$, or $\sqrt[3]{s + e} - \frac{b}{3\sqrt[3]{s + e}}$.

And thus we have found the firſt of the three expreſſions that were ſet down in Art. 2. as being equal to x, or the root of the equation $x^3 + bx = c$.

THE SECOND EXPRESSION OF THE ROOT OF THE EQUATION $x^3 + bx = c$.

Art. VI. THE ſecond of the ſaid three expreſſions may be found as follows.

Since $3gh$ is $= b$, g will be $= \frac{b}{3h}$, and conſequently g^3 will be $= \frac{b^3}{27h^3}$. Therefore $g^3 - h^3$ will be $= \frac{b^3}{27h^3} - h^3$.

But $g^3 - h^3$ is $= c$.

Therefore $\frac{b^3}{27h^3} - h^3$ will alſo be $= c$. And conſequently $\frac{b^3}{27} - h^6$ will be $= ch^3$, and $\frac{b^3}{27}$ will be $= h^6 + ch^3$. Therefore $h^6 + ch^3 + \frac{cc}{4}$ will be $= \frac{b^3}{27} + \frac{cc}{4}$, and conſequently $h^3 + \frac{c}{2}$ will be $= \sqrt{\frac{b^3}{27} + \frac{cc}{4}}$, and h^3 will be $= \sqrt{\frac{b^3}{27} + \frac{cc}{4}} - \frac{c}{2}$, or $-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}$. Therefore h will be $= \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$;

and

and g, or $\frac{b}{3b}$ will be $= \frac{b}{3\sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}}}$.

Therefore $g - h$, or $\frac{b}{3b} - h$, will be $=$

$$\frac{b}{3\sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}}} - \sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}}.$$

Therefore x, which is $= g - h$, will alſo be $=$

$$\frac{b}{3\sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}}} - \sqrt[3]{-\frac{c}{2}+\sqrt{\frac{b^3}{27}+\frac{cc}{4}}};$$ which was the ſecond expreſſion for the value of x ſet down above in Art. 2.

Q. E. I.

If e be ſubſtituted inſtead of $\frac{c}{2}$ and ss inſtead of $\frac{b^3}{27}+\frac{cc}{4}$, and conſequently s inſtead of $\sqrt{\frac{b^3}{27}+\frac{cc}{4}}$, in the laſt expreſſion, it will be thereby converted into $\frac{b}{3\sqrt[3]{-e+s}} - \sqrt[3]{-e+s}$, or $\frac{b}{3\sqrt[3]{s-e}} - \sqrt[3]{s-e}$. Therefore x, or the root of the equation $x^3 + bx = c$, will be $= \frac{b}{3\sqrt[3]{s-e}} - \sqrt[3]{s-e}$.

THE THIRD EXPRESSION OF THE ROOT OF THE EQUATION $x^3 + bx = c$.

Art. VII. The third, or laſt, of the ſaid three expreſſions of the value of x, or of the root of the equation $x^3 + bx = c$, ſet down in Art. 2, to wit, the expreſſion

$\sqrt{}$

$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$, may be found as follows.

Since g is ſhewn in Art. 5 to be $= \sqrt{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$, and b is ſhewn in Art. 6 to be $= \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$, it follows that $g - b$ will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$; that is, x, or the root of the equation $x^3 + bx = c$, will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}} - \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{b^3}{27} + \frac{cc}{4}}}$.

Q. E. I.

And, if e be ſubſtituted inſtead of $\frac{c}{2}$, and s inſtead of $\sqrt{\frac{b^3}{27} + \frac{cc}{4}}$, in this expreſſion, it will be thereby converted into $\sqrt[3]{e + s} - \sqrt[3]{-e + s}$, or $\sqrt[3]{s + e} - \sqrt[3]{s - e}$. Therefore x, or the root of the equation $x^3 + bx = c$, will be $= \sqrt[3]{s + e} - \sqrt[3]{s - e}$.

We will now proceed to demonſtrate the truth of theſe three expreſſions of the value of x ſynthetically, by ſubſtituting each of them inſtead of x in the binomial quantity $x^3 + bx$, and ſhewing that the value of the ſaid binomial quantity reſulting from the ſaid ſubſtitution will be equal to c, or the abſolute term of the equation $x^3 + bx = c$.

A SYNTHETICK DEMONSTRATION

THAT THE EXPRESSION $\sqrt[3]{s+e} - \frac{b}{3\sqrt[3]{s+e}}$, OBTAINED BY THE INVESTIGATION IN ART. 5, IS EQUAL TO THE ROOT OF THE EQUATION $x^3 + bx = c$.

Art. VIII. IF x is $= \sqrt[3]{s+e} - \frac{b}{3\sqrt[3]{s+e}}$, or $\overline{s+e}^{\frac{1}{3}} - \frac{b}{3 \times \overline{s+e}^{\frac{1}{3}}}$, we shall have $x^3 \left(= s + e - 3 \times \overline{s+e}^{\frac{2}{3}} \times \frac{b}{3 \times \overline{s+e}^{\frac{1}{3}}} + 3 \times \overline{s+e}^{\frac{1}{3}} \times \frac{bb}{9 \times \overline{s+e}^{\frac{2}{3}}} - \frac{b^3}{27 \times s+e} = s + e - \frac{\overline{s+e}^{\frac{2}{3}} \times b}{\overline{s+e}^{\frac{1}{3}}} + \frac{\overline{s+e}^{\frac{1}{3}} \times bb}{3 \times \overline{s+e}^{\frac{2}{3}}} - \frac{b^3}{27 \times \overline{s+e}}\right) = s + e - \overline{s+e}^{\frac{1}{3}} \times b + \frac{bb}{3 \times \overline{s+e}^{\frac{1}{3}}} - \frac{b^3}{27 \times \overline{s+e}}$; and we shall have $bx \left(= b \times \overline{s+e}^{\frac{1}{3}} - \frac{b}{3 \times \overline{s+e}^{\frac{1}{3}}}\right) = b \times \overline{s+e}^{\frac{1}{3}} - \frac{bb}{3 \times \overline{s+e}^{\frac{1}{3}}}$.

Therefore $x^3 + bx$ will be $=$

$$\left\{\begin{array}{l} s + e - b \times \overline{s+e}^{\frac{1}{3}} + \frac{bb}{3 \times \overline{s+e}^{\frac{1}{3}}} - \frac{b^3}{27 \times \overline{s+e}} \\ \qquad + b = \overline{s+e}^{\frac{1}{3}} - \frac{bb}{3 \times \overline{s+e}^{\frac{1}{3}}} \end{array}\right\}$$

$= s$

$= s + e \ * \ * \ - \frac{b^3}{27 \times \overline{s + e}} = \frac{27 \times \overline{s+e}^2 - b^3}{27 \times \overline{s + e}} =$
$\frac{27 \times \overline{ss + 2se + ee} - b^3}{27 \times \overline{s + e}} = \frac{27ss + 54se + 27ee - b^3}{27s + 27e}.$

But, becauſe e is $\frac{c}{2}$, we ſhall have $ee = \frac{cc}{4}$; and conſequently ss (which is $= \frac{b^3}{27} + \frac{cc}{4}$,) will be $= \frac{b^3}{27} + ee$. Therefore $\frac{b^3}{27}$ will be $= ss - ee$, and b^3 will be $= 27 \times \overline{ss - ee} = 27ss - 27ee$.

Therefore the fraction $\frac{27ss + 54se + 27ee - b^3}{27s + 27e}$ will be $= \frac{27ss + 54se + 27ee - \sqrt{27ss - 27ee}}{27s + 27e} =$
$\frac{27ss + 54se + 27ee - 27ss + 27ee.}{27s + 27e} = \frac{54se + 54ee}{27s + 27e} =$
$\frac{2se + 2ee}{s + e} = \frac{2e \times s + e}{s + e} = 2e = 2 \times \frac{c}{2} = c.$

Therefore $x^3 + bx$ (which has been ſhewn to be equal to the fraction $\frac{27ss + 54se + 27ee - b^3}{27s + 27e}$) will be $= c$, or the abſolute term of the equation $x^3 + bx = c$; and conſequently $\overline{s + e}^{\frac{1}{3}} - \frac{b}{3 \times \overline{s + e}^{\frac{1}{3}}}$, or $\sqrt[3]{s + e} - \frac{b}{3\sqrt[3]{s + e}}$, will be equal to x, or the root of the ſaid equation.

Q. E. D.

A SYNTHETICK DEMONSTRATION

THAT THE EXPRESSION $\frac{b}{3\sqrt[3]{s-e}} - \sqrt[3]{s-e}$, OBTAINED IN ART. VI, IS EQUAL TO THE ROOT OF THE EQUATION $x^3 + bx = c$.

Art. IX. IF x is equal to $\frac{b}{3\sqrt[3]{s-e}} - \sqrt[3]{s-e}$, or to $\frac{b}{3 \times s-e|^{\frac{1}{3}}} = s-e|^{\frac{1}{3}}$, we shall have $x^3 (= \frac{b^3}{27 \times \sqrt{s-e}}$ $- 3 \times \frac{bb}{9 \times s-e|^{\frac{2}{3}}} \times s-e|^{\frac{2}{3}} + 3 \times \frac{b}{3 \times s-e|^{\frac{1}{3}}} \times$ $s-e|^{\frac{2}{3}} - \sqrt{s-e} = \frac{b^3}{27 \times s-e} - \frac{bb \times s-e|^{\frac{1}{3}}}{3 \times s-e|^{\frac{2}{3}}} +$ $\frac{b \times s-e|^{\frac{2}{3}}}{s-e|^{\frac{1}{3}}} - \sqrt{s-e} = \frac{b^3}{27 \times s-e} - \frac{bb}{3 \times s-e|^{\frac{1}{3}}} +$ $b \times s-e|^{\frac{1}{3}} - \sqrt{s-e}$, and $bx (= b \times$ $\left(\frac{b}{3 \times s-e|^{\frac{1}{3}}} - s-e|^{\frac{1}{3}}\right) = \frac{bb}{3 \times s-e|^{\frac{1}{3}}} - b \times s-e|^{\frac{1}{3}}$.

Therefore $x^3 + bx$ will be $=$

$$\left\{ \begin{array}{l} \frac{b^3}{27 \times s-e} - \frac{bb}{3 \times s-e|^{\frac{1}{3}}} + b \times s-e|^{\frac{1}{3}} - \sqrt{s-e} \\ \qquad + \frac{bb}{3 \times s-e|^{\frac{1}{3}}} - b \times s-e|^{\frac{1}{3}} \end{array} \right\}$$

$= \frac{b^3}{27 \times s - e}$ * * $-\sqrt{s - e} =$

$\frac{b^3 - 27 \times s - e\sqrt{}}{27 \times s - e} = \frac{b^3 - 27 \times \sqrt{ss - 2se + ee}}{27 \times s - e} =$

$\frac{b^3 - 27ss + 54se - 27ee}{27s - 27e}$.

But, becauſe ss is $= \frac{b^3}{27} + \frac{ee}{4} = \frac{b^3}{27} + ee$, we ſhall have $\frac{b^3}{27} = ss - ee$, and conſequently $b^3 = 27ss - 27ee$.

Therefore the fraction $\frac{b^3 - 27ss + 54se - 27ee}{27s - 27e}$ will be

$= \frac{27ss - 27ee - 27ss + 54se - 27ee}{27s - 27e} = \frac{54se - 54ee}{27s - 27e} =$

$\frac{2se - 2ee}{s - e} = 2e = 2 \times \frac{c}{2} = c$.

Therefore $x^3 + bx$ (which has been ſhewn to be equal to the fraction $\frac{b^3 - 27ss + 54se - 27ee}{27s - 27e}$) will be $= c$, and conſequently $\frac{b}{3 \times \sqrt{s - e}^{\frac{1}{3}}} - \sqrt{s - e}^{\frac{1}{3}}$, or $\frac{b}{3\sqrt[3]{s - e}} - \sqrt[3]{s - e}$, will be equal to x, or the root of the equation $x^3 + bx = c$.

Q. E. D.

A SYN-

A SYNTHETICK DEMONSTRATION

THAT THE EXPRESSION $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, OBTAINED IN ART. VII, IS EQUAL TO THE ROOT OF THE EQUATION $x^3 + bx = c$.

Art. X. IF x is $= \sqrt[3]{s+e} - \sqrt[3]{s-e}$, or $\overline{s+e}|^{\frac{1}{3}} - \overline{s-e}|^{\frac{1}{3}}$, we shall have $x^3 = s + e - 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} + 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} - \overline{s-e}$, and $bx \left(= b \times \overline{\overline{s+e}|^{\frac{1}{3}} - \overline{s-e}|^{\frac{1}{3}}}\right) = b \times \overline{s+e}|^{\frac{1}{3}} - b \times \overline{s-e}|^{\frac{1}{3}}$.

Therefore $x^3 + bx$ will be $= s + e - 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} + 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} - \overline{s-e} + b \times \overline{s+e}|^{\frac{1}{3}} - b \times \overline{s-e}|^{\frac{1}{3}}$.

But, becauſe ss is $= \frac{b^3}{27} + \frac{cc}{4}$, or $\frac{b^3}{27} + ee$, we ſhall have $\frac{b^3}{27} = ss - ee$, and conſequently $b^3 = 27 \times \overline{ss - ee} = 27 \times \overline{s+e} \times \overline{s-e}$.

Therefore b will be $= 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{1}{3}}$; and $b \times \overline{s+e}|^{\frac{1}{3}}$ will be $= 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{1}{3}} \times \overline{s+e}|^{\frac{1}{3}} = 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}}$; and $b \times \overline{s-e}|^{\frac{1}{3}}$ will be $= 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{1}{3}} = 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}}$.

Therefore $s + e - 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} + 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} - \overline{s-e} + b \times \overline{s+e}|^{\frac{1}{3}} - b \times \overline{s-e}|^{\frac{1}{3}}$ will be $= s + e - 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} + 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} - \overline{s-e} + 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} - 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} = s + e - \overline{s-e} = s + e - s + e = 2e = 2 \times \frac{c}{2} = c.$

Therefore $x^3 + bx$ (which has been ſhewn to be equal to $s + e - 3 \times \overline{s+e}|^{\frac{2}{3}} \times \overline{s-e}|^{\frac{1}{3}} + 3 \times \overline{s+e}|^{\frac{1}{3}} \times \overline{s-e}|^{\frac{2}{3}} - \overline{s-e} + b \times \overline{s+e}|^{\frac{1}{3}} - b \times \overline{s-e}|^{\frac{1}{3}}$) will be $= c$; and conſequently the expreſſion $\overline{s+e}|^{\frac{1}{3}} - \overline{s-e}|^{\frac{1}{3}}$, or $\sqrt[3]{s+e} - \sqrt[3]{s-e}$, will be equal to x, or the root of the equation $x^3 + bx = c$.

Q. E. D.

THE INVESTIGATION OF

CARDAN's SECOND RULE,

WHICH RELATES TO THE RESOLUTION OF THE EQUATION

$$x^3 - bx = c.$$

Art. XI. PREVIOUSLY to the inveſtigation of this celebrated Rule, it will be proper to make the following obſervations.

Obſervation 1. In the cubick equation $x^3 - bx = c$, (which is a propoſition affirming that x^3, or the cube of the

the unknown quantity x, is greater than bx, or the product of the multiplication of x by the known co-efficient b, and that the excefs, or difference, is equal to the known quantity c,) it is evident that, fince x^3 is greater than bx, $\frac{x^3}{x}$, or xx, muft be greater than $\frac{bx}{x}$, or b, and conſequently that x muft be greater than $\sqrt{b}$.

Obſervation 2. While x increaſes from $\sqrt{b}$ *ad infinitum*, x^3 will increaſe continually from $b \times \sqrt{b}$ *ad infinitum*, and bx will alſo increaſe continually from the ſame quantity $b \times \sqrt{b}$ *ad infinitum*.

Obſervation 3. And, while x increaſes from $\sqrt{b}$ *ad infinitum*, the excefs of x^3 above bx will increaſe continually, or without ever decreaſing, from ($b\sqrt{b} - b\sqrt{b}$, or) 0 *ad infinitum*.

For, if we put $\dot{x}$, (or the letter x with a point placed over it,) to denote the increment which x receives in any given portion of time, either ſmall or great, during its increaſe from $\sqrt{b}$ *ad infinitum*, $b\dot{x}$ will be the increment which bx will receive in the ſame time, and $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ will be the increment which x^3 will receive in the ſame time; becauſe, when x is increaſed to $x + \dot{x}$, bx will be increaſed to $b \times x + \dot{x}$, or to $bx + b\dot{x}$, and x^3 will be increaſed to $x + \dot{x}^3$, or to $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$. Now, ſince xx is always greater than b during the whole increaſe of x from being equal to $\sqrt{b}$ *ad infinitum*, it follows that $x^2 \times \dot{x}$, or $x^2\dot{x}$, will be always greater than $b \times \dot{x}$, or $b\dot{x}$, during that whole increaſe. But $3x^2\dot{x}$ is greater than $x^2\dot{x}$, and $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ is ſtill greater than $3x^2\dot{x}$. Therefore, *à fortiori*, $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ muſt be greater than $b\dot{x}$ during the

whole increaſe of x; that is, the increment of x^3 will be greater than the contemporary increment of bx during the whole increaſe of x. Therefore the reſidual quantity $x^3 - bx$, or the exceſs of x^3 above bx, will increaſe from o continually, or without ever decreaſing, while x increaſes from $\sqrt{b}$ to any greater magnitude.

Further, ſince $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ is the increment of x^3, and $b\dot{x}$ is the contemporary increment of bx, and $x^2\dot{x}$ is greater than $b\dot{x}$, it follows that the exceſs of the increment of x^3 above the contemporary increment of bx will be greater than the exceſs of the increment of x^3 above $x^2\dot{x}$, or than the exceſs of $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ above $x^2\dot{x}$, or than the quantity $(3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3 - x^2\dot{x}$, or) $2x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$. But the exceſs of the increment of x^3 above the contemporary increment of bx is the increment of the reſidual quantity $x^3 - bx$. Therefore the increment of the reſidual quantity $x^3 - bx$ is greater than the quantity $2x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$. But it is evident that the quantity $2x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$ will increaſe continually *ad infinitum*, while x increaſes *ad infinitum*; ſo that no quantity can be aſſigned, how great ſoever, which the ſaid quantity $2x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, or either of its two firſt members $2x^2\dot{x}$ and $3x\dot{x}^2$, (in which x is involved,) may not, by increaſing the quantity x continually, be made to exceed. Therefore the increment of the reſidual quantity $x^3 - bx$ (which increment is greater than the quantity $2x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$,) will increaſe continually *ad infinitum*, or ſo as to become greater than any finite quantity, how great ſoever. And conſequently the reſidual quantity $x^3 - bx$ itſelf (which receives the ſaid continually-increaſing and infinitely-increaſing increment,)

crement,) will increaſe continually from o *ad infinitum*, or ſo as to become greater than any finite quantity, how great ſoever.

Q. E. D.

Obſervation 4. Since the compound quantity $x^3 - bx$ increaſes continually at the ſame time as x increaſes, and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$ the ſaid compound quantity is $(= \frac{8b\sqrt{b}}{3\sqrt{3}} - \frac{2b\sqrt{b}}{\sqrt{3}} = \frac{8b\sqrt{b}}{3\sqrt{3}} - \frac{3 \times 2b\sqrt{b}}{3\sqrt{3}} = \frac{8b\sqrt{b}}{3\sqrt{3}} - \frac{6b\sqrt{b}}{3\sqrt{3}}) = \frac{2b\sqrt{b}}{3\sqrt{3}}$; it follows that, if x is greater than $\frac{2\sqrt{b}}{\sqrt{3}}$, the compound quantity $x^3 - bx$ will be greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$; and, if x is leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$, the ſaid compound quantity will be leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$; and, *è converſo*, if the compound quantity $x^3 - bx$, or, it's equal, the abſolute term c, is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, the value of x will be greater than $\frac{2\sqrt{b}}{\sqrt{3}}$; and, if $x^3 - bx$, or c, is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, the value of x will be leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$; or, if $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, x will be greater than $\frac{2\sqrt{b}}{\sqrt{3}}$; and, if $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$ x, will be leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$.

Obſervation 5. When c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$ or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, and conſequently (by the laſt obſer-

vation) x is greater than $\frac{2\sqrt{b}}{\sqrt{3}}$, xx will be greater than $\frac{4b}{3}$, and consequently $\frac{xx}{4}$ will be greater than $\frac{b}{3}$. But $\frac{xx}{4}$ is the ſquare of $\frac{x}{2}$. Therefore, when c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, the ſquare of $\frac{x}{2}$, or of half the unknown quantity x, will be greater than $\frac{b}{3}$. But, (by Euclid's Elements, Book II, Prop. 5,) it is always poſſible to divide a line, as x, into two unequal parts of ſuch magnitudes that the rectangle under the ſaid parts ſhall be equal to any quantity that is leſs than the ſquare of its half. Therefore, when c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, it is poſſible to divide the line x, or the root of the equation $x^3 - bx = c$, into two unequal parts of ſuch magnitudes that their rectangle, or product, ſhall be equal to $\frac{b}{3}$.

This laſt obſervation is the foundation of Cardan's Second Rule, which relates to the reſolution of the equation $x^3 + bx = c$ in the firſt caſe of it, or when c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$; the inveſtigation of which Rule I ſhall now proceed to explain by the Solution of the following Problem.

PRO-

PROBLEM II.

Art. XII. To reſolve the cubick equation $x^3 - bx = c$, when the abſolute term c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$.

SOLUTION.

SINCE c is ſuppoſed to be greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and conſequently (by Obſervation 5) $\frac{xx}{4}$ is greater than $\frac{b}{3}$, it is poſſible for x to be divided into two unequal parts of ſuch magnitudes that their rectangle, or product, ſhall be equal to $\frac{b}{3}$. Now let it be conceived to be ſo divided; and let the greater of it's two parts be called v, and the leſſer be called z. Then will vz be $= \frac{b}{3}$, and conſequently $3vz$ will be $= b$, and $3vz \times \overline{v + z}$ will be $= b \times \overline{v + z}$.

Now ſince $v + z$ is $= x$, we ſhall have $x^3 = \overline{v + z}^3 = v^3 + 3v^2z + 3vz^2 + z^3 = v^3 + z^3 + 3v^2z + 3vz^2 = v^3 + z^3 + 3vz \times \overline{v + z}$; and we ſhall have $bx = b \times \overline{v + z}$. Therefore $x^3 - bx$ will be $= v^3 + z^3 + 3vz \times \overline{v + z} - b \times \overline{v + z}$; that is, (becauſe $3vz \times \overline{v + z}$ is equal

equal to $b \times \overline{v + z}) x^3 - bx$ will be $= v^3 + z^3$. Therefore the abſolute term c (which is equal to $x^3 - bx$) will be $= v^3 + z^3$.

But, ſince $3vz$ is $= b$, we ſhall have $z = \frac{b}{3v}$, and conſequently $z^3 = \frac{b^3}{27v^3}$. Therefore $v^3 + z^3$ will be $= v^3 + \frac{b^3}{27v^3}$; and conſequently c (which has been ſhewn to be equal to $v^3 + z^3$,) will be $= v^3 + \frac{b^3}{27v^3}$. Therefore (multiplying both ſides of the equation by v^3) we ſhall have $cv^3 = v^6 + \frac{b^3}{27}$, and (ſubtracting v^6 from both ſides,) $cv^3 - v^6 = \frac{b^3}{27}$.

But $cv^3 - v^6$ is the product of the multiplication of $c - v^3$ into v^3, which factors are together equal to c. Therefore (by Euclid's Elements, Book II, Prop. 5) $cv^3 - v^6$ muſt be leſs than the ſquare of half of c, that is, than $\frac{cc}{4}$, and conſequently may be ſubtracted from $\frac{cc}{4}$. Let it be ſo ſubtracted; and let it's equal, $\frac{b^3}{27}$, be alſo ſubtracted from the ſame quantity $\frac{cc}{4}$. And the remainders will be equal to each other; that is, $\frac{cc}{4} - \overline{cv^3 - v^6}$, or $\frac{cc}{4} - cv^3 + v^6$, will be equal to $\frac{cc}{4} - \frac{b^3}{27}$. Therefore the ſquare-root of the trinomial quantity

$\frac{cc}{4}$

$\frac{cc}{4} - cv^3 + v^6$ will be equal to the ſquare root of $\frac{cc}{4} - \frac{b^3}{27}$. But the ſquare-root of the trinomial quantity $\frac{cc}{4} - cv^3 + v^6$ is the difference of the ſimple quantities $\frac{c}{2}$ and v^3, that is, either $\frac{c}{2} - v^3$ or $v^3 - \frac{c}{2}$, according as $\frac{c}{2}$ or v^3 is the greater quantity. But it has been already ſhewn that v^3 and z^3 are together equal to c; and v is ſuppoſed to be greater than z, and conſequently v^3 is greater than z^3. Therefore v^3 muſt be greater than the half of $v^3 + z^3$, and conſequently than the half of c, or than $\frac{c}{2}$. Therefore the difference of $\frac{c}{2}$ and v^3 muſt be $v^3 - \frac{c}{2}$, and not $\frac{c}{2} - v^3$; and conſequently $v^3 - \frac{c}{2}$ muſt be the ſquare root of the trinomial quantity $\frac{cc}{4} - cv^3 + v^6$. We ſhall therefore have $v^3 - \frac{c}{2} = \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$; and conſequently (adding $\frac{c}{2}$ to both ſides,) v^3 will be $= \frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$; and therefore (extracting the cube-roots of both ſides,) v will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$.

But z has been ſhewn to be $= \frac{b}{3v}$.

Therefore $v + z$ will be $(= v + \frac{b}{3v}) =$

$\sqrt[3]{}$

$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \frac{b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}}$; and consequently x, or the root of the cubick equation $x^3 - bx = c$, (which is equal to $v + z$,) will be $=$

$$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \frac{b}{3\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}}.$$

Q. E. I.

Art. XIII. This expreſſion of the value of x in the equation $x^3 - bx = c$ may be rendered more ſimple by ſubſtituting in it the ſingle letter e inſtead of $\frac{c}{2}$, and the ſingle letter s inſtead of $\sqrt{\frac{cc}{4} - \frac{b^3}{27}}$. For then it will be $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$. Therefore, if e be put for half the abſolute term c of the cubick equation $x^3 - bx = c$, and ss be put for $\frac{cc}{4} - \frac{b^3}{27}$, or s be put for the ſquare-root of $\frac{cc}{4} - \frac{b^3}{27}$, the root x of the ſaid equation will be equal to $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$, or to $\overline{e + s}^{\frac{1}{3}} + \frac{b}{3 \times \overline{e + s}^{\frac{1}{3}}}$.

A SYN-

A SYNTHETICK DEMONSTRATION OF THE TRUTH OF THE FOREGOING SOLUTION.

Art. XIV. SINCE this expreſſion $\sqrt[3]{e+\sqrt{s}} + \frac{b}{3\times\sqrt[3]{e+\sqrt{s}}}$ is equal to the root x of the equation $x^3 - bx = c$, it is evident that, if it were ſubſtituted inſtead of x in the compound quantity $x^3 - bx$, (which forms the left-hand ſide of that equation,) it ought to make the ſaid compound quantity be equal to the abſolute term c. And ſo we ſhall find it will, if we make the ſaid ſubſtitution; which may be done in the manner following.

If we ſuppoſe x to be $= \sqrt[3]{e+\sqrt{s}} + \frac{b}{3\times\sqrt[3]{e+\sqrt{s}}}$, we ſhall have $x^3 \left(= e + s + 3\times\sqrt[3]{e+\sqrt{s}}^2 \times \frac{b}{3\times\sqrt[3]{e+\sqrt{s}}} + 3\times\sqrt[3]{e+\sqrt{s}} \times \frac{bb}{9\times\sqrt[3]{e+\sqrt{s}}^2} + \frac{b^3}{27\times(e+\sqrt{s})}\right) = e + s + b\times\sqrt[3]{e+\sqrt{s}} + \frac{bb}{3\times\sqrt[3]{e+\sqrt{s}}} + \frac{b^3}{27\times e+s}$, and $bx \left(= b\times\sqrt[3]{e+\sqrt{s}} + \frac{b}{3\times\sqrt[3]{e+\sqrt{s}}}\right) = b\times\sqrt[3]{e+\sqrt{s}} + \frac{bb}{3\times\sqrt[3]{e+\sqrt{s}}}$, and conſequently $x^3 - bx = e + s + b\times\sqrt[3]{e+\sqrt{s}} + \frac{bb}{3\times\sqrt[3]{e+\sqrt{s}}} + \frac{b^3}{27\times e+s} - b\times\sqrt[3]{e+\sqrt{s}}$ —

$$-\frac{bb}{3\times\overline{e+s}^{\frac{1}{3}}}=e+s+\frac{b^3}{27\times\overline{e+s}}=\frac{27\times\overline{e+s}\times\overline{e+s}}{27\times\overline{e+s}}$$

$$+\frac{b^3}{27\times\overline{e+s}}=\frac{27\times\overline{ee+2es+ss}}{27\times\overline{e+s}}+\frac{b^3}{27\times\overline{e+s}}=$$

$$\frac{27ee+54es+27ss}{27\times\overline{e+s}}+\frac{b^3}{27\times\overline{e+s}}=\frac{27ee+54es+27ss+b^3}{27\times\overline{e+s}}.$$

But, becauſe ss is $=\frac{cc}{4}-\frac{b^3}{27}=ee-\frac{b^3}{27}$, we ſhall have $ss+\frac{b^3}{27}=ee$, and $\frac{b^3}{27}=ee-ss$, and conſequently $b^3=27ee-27ss$.

Therefore the fraction $\frac{27ee+54es+27ss+b^3}{27\times\overline{e+s}}$ will be $=\frac{27ee+54es+27ss+27ee-27ss}{27\times\overline{e+s}}=$

$$\frac{54ee+54es}{27\times\overline{e+s}}=\frac{2ee+2es}{e+s}=\frac{2e\times\overline{e+s}}{e+s}=2e=2\times\frac{c}{2}=c.$$

Therefore the compound quantity x^3-bx (which has been ſhewn to be equal to the ſaid fraction,) will alſo be equal to c. And conſequently $\overline{e+s}^{\frac{1}{3}}+\frac{b}{3\times\overline{e+s}^{\frac{1}{3}}}$ muſt be equal to x, or the root of the equation $x^3-bx=c$.

Q. E. D.

AND-

ANOTHER EXPRESSION FOR THE ROOT OF THE FOREGOING EQUATION $x^3 - bx = c$.

Art. XV. By reſuming the Solution of the foregoing Problem we may find another expreſſion for the root x of the equation $x^3 - bx = c$, to wit, the expreſſion $\sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \frac{b}{3\sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}}$,

or $\sqrt[3]{e - f} + \frac{b}{3\sqrt[3]{e + f}}$, or $\overline{e - f}^{\frac{1}{3}} + \frac{b}{3 \times \overline{e - f}^{\frac{1}{3}}}$.

This expreſſion may be found in the following manner.

THE INVESTIGATION OF THE SAID SECOND EXPRESSION FOR THE VALUE OF THE ROOT x OF THE CUBICK EQUATION $x^3 - bx = c$.

Art. XVI. In Art. 12, we ſuppoſed the line x to be divided into two unequal parts, v and z, of which v was ſuppoſed to be the greater; and we firſt found the value of the greater part v, and then determined the value of the leſſer part z by its relation to v, which is expreſſed by the equation $3vz = b$. But we may with the ſame eaſe firſt determine the value of the leſſer part z, and then derive from the ſaid value of z the value of the greater part v by means

means of the ſame equation $3vz = b$, which expreſſes their relation to each other; whereby we ſhould obtain the ſecond expreſſion of the value of $v + z$, or $z + v$, or x, which was ſet forth in the foregoing article, to wit,

$$\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \frac{b}{3\sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}}, \text{ or}$$

$$\sqrt[3]{e - s} + \frac{b}{3\sqrt[3]{e - s}}, \text{ or } e - s^{\frac{1}{3}} + \frac{b}{3 \times e - s^{\frac{1}{3}}}.$$

This may be done in the manner following.

Since $3vz$ is $= b$, and conſequently v is $= \frac{b}{3z}$, and v^3 is $= \frac{b^3}{27z^3}$; and it has been ſhewn in Art. 12, page 234, that c is $= v^3 + z^3$; it follows that c will be $= \frac{b^3}{27z^3} + z^3$. Therefore, cz^3 will be $= \frac{b^3}{27} + z^6$, and $cz^3 - z^6$ will be $= \frac{b^3}{27}$. Therefore, if we ſubtract both ſides of this equation from $\frac{cc}{4}$ (which is ſuppoſed to be greater than $\frac{b^3}{27}$, and conſequently muſt alſo be greater than, it's equal, $cz^3 - z^6$,) the remainders will be equal to each other; that is, $\frac{cc}{4} - \sqrt{cz^3 - z^6}$, or $\frac{cc}{4} - cz^3 + z^6$, will be $= \frac{cc}{4} - \frac{b^3}{27}$. Therefore the ſquare-root of the trinomial quantity $\frac{cc}{4} - cz^3 + z^6$ will be equal to the ſquare-root of $\frac{cc}{4} - \frac{b^3}{27}$. But the ſquare-root of the trinomial quantity

$\frac{cc}{4} - cz^3 + z^6$ is the difference of the ſimple quantities $\frac{c}{2}$ and z^3, that is, either $\frac{c}{2} - z^3$, or $z^3 - \frac{c}{2}$, according as $\frac{c}{2}$ or z^3 is the greater quantity. But, becauſe c is $= v^3 + z^3$, and v^3 is greater than z^3, it follows that z^3 muſt be leſs than one-half of $v^3 + z^3$, or than one-half of c, or than $\frac{c}{2}$; and conſequently the difference between $\frac{c}{2}$ and z^3 will be $\frac{c}{2} - z^3$, and not $z^3 - \frac{c}{2}$. Therefore $\frac{c}{2} - z^3$ will be the ſquare-root of the trinomial $\frac{cc}{4} - cz^3 + z^6$ and conſequently will be equal to the ſquare-root of $\frac{cc}{4} - \frac{b^3}{27}$, or $\frac{c}{2} - z^3$ will be $= \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$. Therefore (adding z^3 to both ſides,) $\frac{c}{2}$ will be $= z^3 + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$, and (ſubtracting $\sqrt{\frac{cc}{4} - \frac{b^3}{27}}$ from both ſides,) z^3 will be $= \frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$, and conſequently (extracting the cube-roots of both ſides,) z will be $= \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$. Therefore $z + v$ will be $= \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + v = \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \frac{b}{3z} = \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$

$+\frac{b}{3\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}}$. Therefore x (which is equal to $z+v$) will alſo be $=\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}+\frac{b}{3\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}}$; that is, the root of the equation $x^3-bx=c$ will be $=\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}+\frac{b}{3\sqrt[3]{\frac{c}{2}-\sqrt{\frac{cc}{4}-\frac{b^3}{27}}}}$, or $\sqrt[3]{e-s}+\frac{b}{3\sqrt[3]{e-s}}$, or $\overline{e-s}\big|^{\frac{1}{3}}+\frac{b}{3\times\overline{e-s}\big|^{\frac{1}{3}}}$.

Q. E. I.

A SYNTHETICK DEMONSTRATION OF THE TRUTH OF THE FOREGOING EXPRESSION.

Art. XVII. Here again we may demonſtrate ſynthetically, that this expreſſion $\overline{e-s}\big|^{\frac{1}{3}}+\frac{b}{3\times\overline{e-s}\big|^{\frac{1}{3}}}$ is equal to the true value of x in the propoſed equation $x^3-bx=c$, by ſubſtituting the ſaid expreſſion for x in the compound quantity x^3-bx, which forms the left-hand ſide of the ſaid equation. For, if we make this ſubſtitution,

tion, we ſhall find that the value of $x^3 - bx$ thence ariſing will be equal to the abſolute term c. This may be ſhewn in the manner following.

If x is ſuppoſed to be equal to $\sqrt[3]{e-s} + \frac{b}{3\times\sqrt[3]{e-s}}$, we ſhall have $x^3 = e - s + 3 \times \sqrt[3]{e-s}^2 \times \frac{b}{3\times\sqrt[3]{e-s}} + 3\times\sqrt[3]{e-s}\times\frac{bb}{9\times\sqrt[3]{e-s}^2} + \frac{b^3}{27\times(e-s}$ $= e - s + b \times \sqrt[3]{e-s}^2 + \frac{bb}{3\times\sqrt[3]{e-s}} + \frac{b^3}{27\times(e-s}$ and $bx \left(= b \times \sqrt[3]{e-s} + \frac{b}{3\times\sqrt[3]{e-s}}\right) = b\times\sqrt[3]{e-s}^2 + \frac{bb}{3\times\sqrt[3]{e-s}}$, and conſequently $x^3 - bx =$

$$\left\{\begin{array}{r} e - s + b\times\sqrt[3]{e-s}^2 + \frac{bb}{3\times\sqrt[3]{e-s}} + \frac{b^3}{27\times e-s} \\ - b\times\sqrt[3]{e-s}^2 - \frac{bb}{3\times\sqrt[3]{e-s}} \end{array}\right\}$$

$= e - s \quad * \quad * \quad + \frac{b^3}{27\times e-s}$

$= \frac{27\times\sqrt{e-s}\times\sqrt{e-s}}{27\times e-s} + \frac{b^3}{27\times e-s} = \frac{27\times ee - 2es + ss}{27\times e-s}$

$+ \frac{b^3}{27\times e-s} = \frac{27ee - 54es + 27ss}{27\times e-s} + \frac{b^3}{27\times e-s} =$

$\frac{27ee-54es+27ss+b^3}{27\times e-s}$.

But it has been ſhewn in Art. 14, that b^3 is $= 27e^2 - 27s^2$.

Therefore the fraction $\frac{27ee - 54es + 27ss + b^3}{27 \times e - s}$ is $= \frac{27ee - 54es + 27ss + 27ee - 27ss}{27 \times e - s} = \frac{54ee - 54es}{27 \times e - s} = \frac{2ee - 2es}{e - s} = \frac{2e \times e - s}{e - s} = 2e = 2 \times \frac{c}{2} = c$. And consequently the compound quantity $x^3 - bx$ (which has been shewn to be equal to the said fraction,) will also be equal to c; that is, the value of the compound quantity $x^3 - bx$ arising from the substitution of $\sqrt[3]{e - s} + \frac{b}{3 \times \sqrt[3]{e - s}}$ in it's terms instead of x, is equal to the absolute term c of the cubick equation $x^3 - bx = c$. Therefore $\sqrt[3]{e - s} + \frac{b}{3 \times \sqrt[3]{e - s}}$ must be equal to the value of x in the said equation.

Q. E. D.

A THIRD EXPRESSION FOR THE ROOT OF THE FOREGOING EQUATION $x^3 - bx = c$.

Art. XVIII. We may also, by resuming the Solution of the foregoing Problem contained in Art. 12, find a third expression for the root of this equation $x^3 - bx = c$, to wit, the expression $\sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} +$

$\sqrt[3]{}$

$\sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$ or $\sqrt[3]{e + s} + \sqrt[3]{e - s}$, or $\overline{e + s}^{\frac{1}{3}} + \overline{e - s}^{\frac{1}{3}}$. This expreſſion may be found in the following manner.

THE INVESTIGATION OF THE SAID THIRD EXPRESSION FOR THE VALUE OF THE ROOT x OF THE CUBICK EQUATION $x^3 - bx = c$.

Art. XIX. SINCE $v^3 + z^3$ is $= c$, it follows that z^3 will be $= c - v^3$. But v^3 is ſhewn, in Art. 12, to be $= \frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$. Therefore $c - v^3$ will be $c - \frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}} = c - \frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}} = \frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$. Conſequently z^3 (which is $= c - v^3$,) will be $= \frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$; and therefore z will be $= \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$.

But v^3 has been ſhewn in Art. 12, to be $= \frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}$, and conſequently v is $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$.

Therefore $v + z$ will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$.

But x is $= v + z$.

Therefore x, or the root of the equation $x^3 - bx = c$, will be $= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{cc}{4} - \frac{b^3}{27}}} + \sqrt[3]{\frac{c}{2} - \sqrt{\frac{cc}{4} - \frac{b^3}{27}}}$, or $= \sqrt[3]{e + s} + \sqrt[3]{e - s}$, or $= \overline{e + s}^{\frac{1}{3}} + \overline{e - s}^{\frac{1}{3}}$.

Q. E. I.

A SYNTHETICK DEMONSTRATION OF THE TRUTH OF THE FOREGOING EXPRESSION.

Art. XX. HERE again we may demonſtrate ſynthetically, that this expreſſion $\overline{e + s}^{\frac{1}{3}} + \overline{e - s}^{\frac{1}{3}}$ is equal to the true value of x in the propoſed equation $x^3 - bx = c$, by ſubſtituting it for x in the compound quantity $x^3 - bx$, which forms the firſt, or the left-hand, ſide of the ſaid equation. For, if we make this ſubſtitution, we ſhall find that the value of $x^3 - bx$ thence ariſing will be equal to the abſolute term c of the ſaid equation. This may be ſhewn in the following manner:

If

If x is ſuppoſed to be $= \sqrt[3]{e + s} + \sqrt[3]{e - s}$, we ſhall have $x^3 = e + s + 3 \times \sqrt[3]{e + s}^2 \times \sqrt[3]{e - s} + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{e - s}^2 + e - s = 2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} \times \sqrt[3]{e - s} = 2e + 3 \times \sqrt[3]{e + s}^2 \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} + 3 \times \sqrt[3]{e - s}^2 \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} = 2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} \times \sqrt[3]{e - s} + 3 \times \sqrt[3]{e - s} \times \sqrt[3]{e + s} \times \sqrt[3]{e - s} = 2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss} + 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss}$; and we ſhall have $bx = b \times \sqrt[3]{e + s} + \sqrt[3]{e - s} = b \times \sqrt[3]{e + s} + b \times \sqrt[3]{e - s}$; and conſequently $x^3 - bx$ will be $= 2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss} + 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss} - b \times \sqrt[3]{e + s} - b \times \sqrt[3]{e - s}$.

But it has been ſhewn in Art. 14, that b^3 is $= 27ee - 27ss = 27 \times \sqrt{ee - ss}$. Therefore b will be $= 3 \times \sqrt[3]{ee - ss}$. And conſequently $b \times \sqrt[3]{e + s}$ will be $= 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e + s} = 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss}$; and $b \times \sqrt[3]{e - s}$ will be $= 3 \times \sqrt[3]{ee - ss} \times \sqrt[3]{e - s} = 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss}$.

Therefore $x^3 - bx$ (which is equal to $2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss} + 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss} - b \times \sqrt[3]{e + s} - b \times \sqrt[3]{e - s}$,) will be $=$

$$\left\{\begin{array}{l} 2e + 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss} + 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss} \\ \quad - 3 \times \sqrt[3]{e + s} \times \sqrt[3]{ee - ss} - 3 \times \sqrt[3]{e - s} \times \sqrt[3]{ee - ss} \end{array}\right\}$$

$= 2e$ * *

But e is $= \frac{c}{2}$, and consequently $2e$ is $= c$.

Therefore $x^3 - bx$ will be $= c$; that is, the value of the compound quantity $x^3 - bx$ arising from the substitution of the expression $\overline{e + s^{\frac{1}{3}}} + \overline{e - s^{\frac{1}{3}}}$ instead of x, is equal to the absolute term c of the cubick equation $x^3 - bx = c$. And consequently the said expression $\overline{e + s^{\frac{1}{3}}} + \overline{e - s^{\frac{1}{3}}}$ must be equal to the root x of the said equation.

Q. E. D.

Art. XXI. I do not remember to have seen these substitutions, or synthetick demonstrations of the truth of the expressions given by the foregoing rule of Cardan for the root of the cubick equation $x^3 - bx = c$, nor the former substitutions, or synthetick demonstrations of the truth of the three former expressions given by the first rule of Cardan for the root of the cubick equation $x^3 + bx = c$, in any former book of Algebra.

Art. XXII. I will now, by way of illustration of the expressions that have been above investigated, subjoin an example of the resolution of a numeral equation of the foregoing form, $x^3 - bx = c$, by each of the three expressions above-mentioned; by which it will appear that they will, all three, bring out the same number for the root of the proposed equation.

AN

AN EXAMPLE OF THE RESOLUTION OF A CUBICK EQUATION OF THE FOREGOING FORM, $x^3 - bx = c$, BY MEANS OF EACH OF THE THREE FOREGOING EXPRESSIONS.

Art. XXIII. LET it be required to find the value of x in the equation $x^3 - 3x = 18$.

In this equation b is $= 3$, and c is $= 18$. Therefore $\sqrt{b}$ is $= \sqrt{3}$, and $\frac{2b\sqrt{b}}{3\sqrt{3}}$ is $= \frac{2 \times 3\sqrt{3}}{3\sqrt{3}} = 2$; which is much leſs than 18, or c. Therefore this equation comes under the firſt caſe of the general equation $x^3 - bx = c$, and conſequently may be reſolved by Cardan's ſecond rule, or the rule diſcovered by Nicholas Tartaglia, or by computing either of the three foregoing expreſſions.

THE RESOLUTION OF THE EQUATION $x^3 - 3x = 18$ BY MEANS OF THE FIRST OF THE THREE FOREGOING EXPRESSIONS, TO WIT, THE EXPRESSION $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$.

Art. XXIV. Now, ſince b is $= 3$, $\frac{b}{3}$ will be $= \frac{3}{3} = 1$, and conſequently $\frac{b^3}{27}$, or the cube of $\frac{b}{3}$, will

will alſo be = 1. And, ſince c is = 18, we ſhall have $\frac{c}{2}$, or e, = 9, and $\frac{cc}{4}$, or ee, = 81; and conſequently $\frac{cc}{4} - \frac{b^3}{27}$ will be (= 81 — 1) = 80; that is, ss will be = 80. Therefore s will be (= $\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5}$) = $4 \times \sqrt{5}$, and $e + s$ will be = $9 + 4 \times \sqrt{5}$ (= $\frac{9 \times 8}{8} + \frac{8 \times 4 \times \sqrt{5}}{8} = \frac{72}{8} + \frac{32 \times \sqrt{5}}{8}$ = $\frac{72 + 32 \times \sqrt{5}}{8} = \frac{27 + 27 \times \sqrt{5} + 45 + 5 \times \sqrt{5}}{8}$ = $\frac{3^3 + 3 \times 9 \times \sqrt{5} + 3 \times 3 \times 5 + 5 \times \sqrt{5}}{8}$ = $\frac{3^3 + 3 \times 3^2 \times \sqrt{5} + 3 \times 3 \times 5 + 5 \times \sqrt{5}}{8}$) = $\frac{\overline{3 + \sqrt{5}}^3}{2^3}$; and conſequently $\sqrt[3]{e + s}$ will be = $\frac{3 + \sqrt{5}}{2}$. Therefore $3 \times \sqrt[3]{e + s}$ will be = $3 \times \frac{3 + \sqrt{5}}{2}$, and $\frac{b}{3 \times \sqrt[3]{e + s}}$ will be = $\frac{3}{3 \times \frac{3 + \sqrt{5}}{2}}$ (= $\frac{1}{\frac{3 + \sqrt{5}}{2}} = 1 \times \frac{2}{3 + \sqrt{5}}$) = $\frac{2}{3 + \sqrt{5}}$; and conſequently $\sqrt[3]{e + s} + \frac{b}{3 \times \sqrt[3]{s + e}}$ will be = $\frac{3 + \sqrt{5}}{2} + \frac{2}{3 + \sqrt{5}}$ (= $\frac{\overline{3 + \sqrt{5}} \times \overline{3 + \sqrt{5}}}{2 \times \overline{3 + \sqrt{5}}} + \frac{4}{2 \times 3 + \sqrt{5}} = \frac{9 + 6 \times \sqrt{5} + 5}{2 \times \overline{3 + \sqrt{5}}}$ + $\frac{4}{2 \times \overline{3 + \sqrt{5}}} = \frac{14 + 6\sqrt{5}}{2 \times 3 + \sqrt{5}} + \frac{4}{2 \times 3 + \sqrt{5}} = \frac{18 + 6\sqrt{5}}{2 \times 3 + \sqrt{5}}$ = $\frac{9 + 3\sqrt{5}}{3 + \sqrt{5}}$) = $\frac{3 \times \overline{3 + \sqrt{5}}}{3 + \sqrt{5}} = 3$. Therefore 3 is the value

value of x in the equation $x^3 - 3x = 18$ obtained by means of the expreſſion $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$.

Q. E. I.

Art. XXV. And, upon trial, we ſhall find 3 to be the true value of x in the ſaid equation $x^3 - 3x = 18$. For, if x be ſuppoſed to be $= 3$, we ſhall have $x^3 = 27$, and $3x = 3 \times 3 = 9$, and conſequently $x^3 - 3x = 27 - 9 = 18$. Therefore 3 is the true value of x in the ſaid equation.

Q. E. D.

And thus we ſee that the firſt of the three foregoing expreſſions, to wit, the expreſſion $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$, has given us the true value of x in the propoſed equation $x^3 - 3x = 18$.

THE RESOLUTION OF THE SAME EQUATION $x^3 - 3x = 18$ BY MEANS OF THE SECOND OF THE FOREGOING EXPRESSIONS, TO WIT, THE EXPRESSION $\sqrt[3]{e - s} + \frac{b}{3\sqrt[3]{e - s}}$.

Art. XXVI. Now, ſince c is $= 18$, and conſequently $\frac{c}{2}$, or e, is $= 9$, and s has been ſhewn to be $= \sqrt{80}$, or $4\sqrt{5}$,

$4\sqrt{5}$, we ſhall have $e - s = 9 - 4\sqrt{5} = \frac{72 - 32\sqrt{5}}{8}$ $= \frac{27 - 27\sqrt{5} + 45 - 5\sqrt{5}}{8}$, and conſequently $\sqrt[3]{e - s}$ $\frac{3 - \sqrt{5}}{2}$. Therefore $\frac{b}{3\sqrt[3]{e - s}}$ will be $= \frac{3}{3 \times \frac{3 - \sqrt{5}}{2}}$ $(= \frac{1}{\frac{3 - \sqrt{5}}{2}} = 1 \times \frac{2}{3 - \sqrt{5}} = \frac{2}{3 - \sqrt{5}}$; and conſequently $\sqrt[3]{e - s} + \frac{b}{3\sqrt[3]{e - s}}$ will be $= \frac{3 - \sqrt{5}}{2} + \frac{2}{3 - \sqrt{5}}$ $(= \frac{\overline{3 - \sqrt{5}} \times \overline{3 - \sqrt{5}}}{2 \times \overline{3 - \sqrt{5}}} + \frac{4}{2 \times \overline{3 - \sqrt{5}}} =$ $\frac{9 - 6\sqrt{5} + 5}{2 \times \overline{3 - \sqrt{5}}} + \frac{4}{2 \times 3 - \sqrt{5}} = \frac{14 - 6\sqrt{5}}{2 \times \overline{3 - \sqrt{5}}} + \frac{4}{2 \times 3 - \sqrt{5}}$ $= \frac{14 - 6\sqrt{5} + 4}{2 \times 3 - \sqrt{5}} = \frac{18 - 6\sqrt{5}}{2 \times 3 - \sqrt{5}} = \frac{9 - 3\sqrt{5}}{3 - \sqrt{5}}) = \frac{3 \times \overline{3 - \sqrt{5}}}{3 - \sqrt{5}}$ $= 3$. Therefore, according to this ſecond expreſſion $\sqrt[3]{e - s} + \frac{b}{3\sqrt[3]{e - s}}$, the root of the cubick equation $x^3 - 3x = 18$ is 3; as it was found to be by the firſt expreſſion $\sqrt[3]{e + s} + \frac{b}{3\sqrt[3]{e + s}}$.

Q. E. I.

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THE RESOLUTION OF THE SAME EQUATION $x^3 - 3x = 18$ BY MEANS OF THE THIRD, OR LAST, OF THE THREE FOREGOING EXPRESSIONS, TO WIT, THE EXPRESSION $\sqrt[3]{e+s} + \sqrt[3]{e-s}$.

Art. XXVII. SINCE $\sqrt[3]{e+s}$ has been ſhewn, in Art. 24, to be $= \frac{3+\sqrt{5}}{2}$, and $\sqrt[3]{e-s}$ has been ſhewn, in Art. 26, to be $= \frac{3-\sqrt{5}}{2}$, it follows that $\sqrt[3]{e+s} + \sqrt[3]{e-s}$ muſt be $= \frac{3+\sqrt{5}}{2} + \frac{3-\sqrt{5}}{2}$ $= \frac{3+\sqrt{5}+3-\sqrt{5}}{2} = \frac{3+3}{2} = 3$. Therefore by this expreſſion, as well by both the former, the value of x in the equation $x^3 - 3x = 18$ comes out to be 3.

A SCHOLIUM.

Art. XXVIII. I HAVE dwelt the longer on the foregoing explanation of Cardan's rule for reſolving the cubick equation $x^3 - bx = c$ in the firſt caſe of the ſaid equation, or when c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, becauſe I have obſerved that it has been delivered by

by many writers of Algebra, even of the greateſt eminence, with an uncommon degree of obſcurity, and has been made the ſubject of much myſterious and fantaſtick reaſoning, (or, perhaps, I ought rather to ſay, *diſcourſing*, ſince it deſerves not to be called *reaſoning*,) concerning negative and impoſſible quantities *. But now, I hope, it will be found intelligible, and even eaſy, by every reader who is acquainted with the firſt rudiments of Algebra.

Art. XXIX. This method of reſolving cubick equations of this form $x^3 - bx = c$, in the firſt caſe of it, or when

* See Newton's *Arithmetica Univerſalis*, the 2nd Edition, in the year 1722, pages 279, 280, 281; and *Mac Laurin's Algebra*, Part II, Chapter viii. pages 223, 224, and 225, and Part I, Chap. xiv, the Supplement, pages 127, 128, 129, and 130; and *Clairaut's Elements d'Algébre*, pages 282, 283, &c. - - - - - - 297; and *Saunderſon's Algebra*, Vol. II, pages 692, 693, 694, 695, &c. - - - - 707. All theſe writers have, in the places here referred-to, treated this ſubject with an aſtoniſhing degree of obſcurity, and almoſt as if they had been contending with each other which ſhould treat it moſt obſcurely, without, at the ſame time, advancing any abſolutely-falſe poſition, or any thing that was intirely unintelligible. And, if they had been engaged in a contention of this ſort, I think (ſtrange as it may ſeem to many perſons,) that the three pages here referred-to in Sir Iſaac Newton's *Arithmetica Univerſalis* would have intitled him to the prize due to the higheſt degree of obſcurity. How much is it to be lamented that men of ſuch exalted abilities ſhould, by the manner in which they treat theſe difficult ſubjects, ſo much increaſe their difficulty as to exclude the generality of even the ſtudious part of mankind, who are fond of cultivating theſe ſciences, from the pleaſure of underſtanding their writings! When Sir Iſaac Newton firſt publiſhed his great work called *The Principia*, in the year 1687, I believe there were not a dozen perſons

when $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, or c is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and the method, above explained in the firſt part of this Appendix, of reſolving cubick equations of the form $x^3 + bx = c$ in all poſſible caſes of that form, or whatever be the relative magnitudes of the co-efficient b and the abſolute term c, are uſually known by the name of *Cardan's rules*, becauſe they were firſt publiſhed by *Cardan* in his treatiſe of Algebra intitled, *Ars magna quam vulgò* COSSAM *vocant*, ſeu REGULAS ALGEBRAICAS, in A. D. 1545, and not becauſe they were of his invention. For the rule for reſolving cubick equations of the form $x^3 + bx = c$ was firſt diſcovered (as Cardan himſelf informs us,) by one *Scipio Ferreus*, or *Ferreo*, of Bononia, or Bologna, in Italy, about 30 years before the publication of Cardan's treatiſe above-mentioned; and the other rule for reſolving cubick equations of the ſecond form $x^3 - bx = c$, in the firſt caſe

ſons in all Europe who even pretended to underſtand it. He lived forty years after this edition: and ſurely it would have been no improper employment of four or five of theſe forty years, if he had devoted them to the writing that profound treatiſe, or at leaſt ſome of the moſt important parts of it, over again, ſo as to make it intelligible to a greater number of readers, by adopting the accurate and perſpicuous ſtyle in which the celebrated Mr. Huygens had written his excellent treatiſe on Pendulum-Clocks, called *Horologium Oſcillatorium*, which was publiſhed in the year 1673, only 14 years before the firſt edition of the *Principia*. That work was written on a ſubject of rational mechanics that bore a great reſemblance to ſome of the ſubjects treated of in the *Principia*, and it was intelligible to all the mathematicians of the time, and was generally approved and admired by them, and has been mentioned by Sir Iſaac Newton himſelf by the appellation of *eximium libellum*, which it highly deſerves, being, in truth, a model of good writing upon ſubjects of Mathematical Philoſophy.

of

of thoſe equations, or when $\frac{cc}{4}$ is greater than $\frac{b^3}{27}$, or t is greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, (and which has been the ſubject of the latter part of the preſent diſcourſe,) was invented by another Italian named *Nicholas Tartaglia*, or *Tartalea*, who was Profeſſor of Mathematicks at Venice, and diſtinguiſhed himſelf very much by his ingenious diſcoveries. He was a co-temporary and a friend of Cardan, and communicated this invention to him under a ſtrict promiſe that he would keep it ſecret: and, when Cardan afterwards publiſhed it in the treatiſe above-mentioned, Tartalea complained bitterly of his breach of promiſe, and would never afterwards be reconciled to him. Tartalea died in the year 1557. See *Montucla's Hiſtoire des Mathématiques*, Vol. I, pages 462, and 479, 480, and 481, and Dr. Hutton's Mathematical Dictionary.

END of the Inveſtigation and Illuſtration of Cardan's Rules for reſolving the cubick equation $x^3 + bx = c$ *in all poſſible caſes, or in all poſſible magnitudes of the known quantities* b *and* c, *and for reſolving the cubick equation* $x^3 - bx = c$ *in the firſt caſe of it, or when* c *is greater than* $\frac{2b\sqrt{b}}{3\sqrt{3}}$, *or* $\frac{cc}{4}$ *is greater than* $\frac{b^3}{27}$.

OF

OF THE SECOND CASE OF THE CUBICK EQUATION $x^3 - bx = c$, IN WHICH THE ABSOLUTE TERM c IS LESS THAN $\frac{2b \times \sqrt{b}}{3 \times \sqrt{3}}$, OR $\frac{cc}{4}$ IS LESS THAN $\frac{b^3}{27}$.

Art. XXX. WE will now proceed to conſider the other caſe of the cubick equation $x^3 - bx = c$, in which c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$. This caſe (as we have already ſeen), cannot be reſolved by the aforeſaid ſecond rule of Cardan, (which was invented by *Tartaglia*,) becauſe it is impoſſible in this caſe to divide the line, or root, x into two ſuch parts, v and z, that the product or rectangle, under the ſaid parts, to wit, the product vz, ſhall be equal to $\frac{b}{3}$, and conſequently that $3vz$ ſhall be equal to b, and that $3vz \times \overline{v + z}$ ſhall be equal to $b \times \overline{v + z}$; which is a fundamental ſtep in the ſolution of Problem II. given above in Art. 12. And on this account this caſe of the equation $x^3 - bx = c$ has obtained amongſt Algebräiſts the name of *the irreducible caſe*; and particularly amongſt the French writers of Algebra. Monſieur Montucla in his *Hiſtoire des Mathématiques*, Tom. 1, page 487, ſpeaks of it in theſe words: *On doit à Cardan la remarque de la limitation d'un cas des équations cubiques, où il arrive que l'extraction de la racine quarrée qui entre dans la formule, n'eſt pas poſſible. C'eſt ce que nous appellons* le cas irréductible; *dont la difficulté a donné, et donne*

 encore,

encore, la torture aux Analyſtes. It may, however, be always reſolved by means of a tranſcendental * expreſſion, which may be derived from one or other of Cardan's two rules, above-explained, by the help of Sir Iſaac Newton's binomial and reſidual theorems. But the reaſonings neceſſary to be uſed in order to obtain theſe tranſcendental expreſſions are extremely ſubtle and difficult; as may be ſeen from the two long diſcourſes in which they are ſet forth and explained in the ſecond volume of the *Scriptores Logarithmici*, which begin in page 381, and extend to page 575, and yet, I will venture to ſay, are incapable of being made ſhorter by a ſingle page without increaſing the difficulty of underſtanding them. And further, when thoſe tranſcendental expreſſions have been juſtly obtained, and the manner of obtaining them has been perfectly underſtood, the application of them to the reſolution of the cubick equation $x^3 - bx = c$ in this ſecond caſe of it will be found to be very tedious and laborious. I therefore conſider thoſe expreſſions as matters of curioſity, fit for the contemplation of learned and ſcientifick Algebraiſts, rather than as a convenient, practical method of reſolving the ſaid equation $x^3 - bx = c$, or computing the value of its root x, in numeral equations of this form that occur in the courſe of any mathematical reſearches. But it fortunately happens that in this caſe of the ſaid cubick equation the value of its root x lies within narrow limits, and conſequently may al-

* By this phraſe we are to underſtand an expreſſion containing an infinite ſeries of decreaſing terms, and which therefore goes beyond the bounds of common, or finite, geometry, and has, from that circumſtance, been called by the learned Mr. Leibnitz and his followers a *tranſcendental* expreſſion.

ways

ways be obtained with very little difficulty by Mr. Raphſon's method of approximation. The manner of doing this will be the ſubject of the following Problem.

PROBLEM III.

Art. XXXI. TO reſolve the cubick equation $x^3 - bx = c$ by approximation, in the ſecond caſe of the ſaid equation, or when the abſolute term c is leſs than $\frac{ab\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$.

SOLUTION.

In this caſe of the equation $x^3 - bx = c$ the value of the root x is always greater than $\sqrt{b}$, but leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$, (or $\frac{2}{\sqrt{3}} \times \sqrt{b}$, or $\frac{2.000}{1.732 \&c.} \times \sqrt{b}$,) or, nearly, $\frac{2000}{1732} \times \sqrt{b}$. Now the difference of theſe two limits is $\frac{2000}{1732} \times \sqrt{b} - \sqrt{b}$, (or $\frac{2000}{1732} \times \sqrt{b} - \frac{1732}{1732} \times \sqrt{b}$,) or $\frac{268}{1732} \times \sqrt{b}$, which is leſs than a ſixth part of the lower limit $\sqrt{b}$, and is leſs than a ſeventh part of the greater limit $\frac{2000}{1732} \times \sqrt{b}$;

$\times \sqrt{b}$; for 6 times $\frac{268}{1732} \times \sqrt{b}$ is only equal to $\frac{1608}{1732} \times \sqrt{b}$, which is leſs than $\frac{1732}{1732} \times \sqrt{b}$, or than $\sqrt{b}$; and 7 times $\frac{268}{1732} \times \sqrt{b}$ is only equal to $\frac{1876}{1732} \times \sqrt{b}$, which is leſs than $\frac{2000}{1732} \times \sqrt{b}$. Therefore, if we were to begin our approximation from either of theſe limits, by denoting the ſaid limit by the letter a, and the unknown difference between a and x, or the root of the equation $x^3 - bx = c$, by the letter z, and ſubſtituting $a + z$ inſtead of x, if a denoted the leſſer limit $\sqrt{b}$, or $a - z$ inſtead of x, if a denoted the greater limit $\frac{2000}{1732} \times \sqrt{b}$, or $\frac{2\sqrt{b}}{\sqrt{3}}$, in the terms of the equation $x^3 - bx = c$, and then reſolving the new equation thence ariſing, in the ſame manner as if it were a mere ſimple equation, by neglecting all the terms of it that involve in them either the cube or the ſquare of z, agreeably to Mr. Raphſon's directions, we ſhould obtain a tolerably near value of z, and conſequently of $a + z$, or $a - z$, or x, even by this firſt proceſs of the approximation. And conſequently, if, inſtead of taking one of theſe limits for a, or our firſt near value of x, or the baſis of our approximation, we were to take ſome intermediate quantity between $\sqrt{b}$ and $\frac{2\sqrt{b}}{\sqrt{3}}$, or $\frac{2000}{1732} \times \sqrt{b}$, for a, or our firſt near value of x, (which it will always be eaſy to do to great advantage by a conjecture founded on a compariſon between the magnitude of c, or the abſolute term of the equation $x^3 - bx = c$, and $\frac{2b\sqrt{b}}{3\sqrt{3}}$, which is the higheſt limit of its magnitude,) and were then to ſubſtitute

ſtitute the ſaid quantity a, ſo taken by conjecture, inſtead of x in the compound quantity $x^3 - bx$, in order to diſcover whether the value of the ſaid compound quantity reſulting from ſuch ſubſtitution is greater, or leſs, than the abſolute term c, and conſequently whether the ſaid quantity a is greater or leſs than the true value of x, and, having made this diſcovery, we were to ſubſtitute $a+z$ or $a-z$, according as a is leſs or greater than x, inſtead of x in the equation $x^3 - bx = c$, and, omitting (according to Mr. Raphſon's directions,) all the terms that involve either the cube or the ſquare of z, we were then to reſolve the new equation, reſulting from ſuch ſubſtitution, as if it were a mere ſimple equation, the value of z thereby obtained would be nearer to the truth than in the former caſe, when we begun our approximation only from one of the two limits $\sqrt{b}$ and $\frac{2}{\sqrt{3}} \times \sqrt{b}$; and conſequently the value of $a+z$, or $a-z$, or x, obtained by this way of proceeding, would be nearer to the true value of it than in the former caſe. And then a ſecond or a third approximation, grounded on the ſame principles, and conducted in the ſame manner, as the firſt, would give us the value of x, or the root of the propoſed equation $x^3 - bx = c$, to as great a degree of exactneſs as we need deſire.

Q. E. I.

A few examples of this method of reſolving equations of this kind will eaſily ſhew the uſe of it.

EXAMPLES OF THE RESOLUTION OF A CUBICK EQUATION OF THE SECOND FORM $x^3 - bx = c$, IN THE SECOND CASE OF THE SAID FORM, OR WHEN c IS LESS THAN $\frac{2b\sqrt{b}}{3\sqrt{3}}$, OR $\frac{cc}{4}$ IS LESS THAN $\frac{b^3}{27}$,

BY MR. RAPHSON'S METHOD OF APPROXIMATION.

EXAMPLE I.

Art. XXXII. LET it be required to resolve the equation $x^3 - 50x = 120$ by the foregoing method of approximation.

Here b is $= 50$; c is $= 120$; $\frac{c}{2}$ is $= 60$; $\frac{cc}{4}$ is $= 3600$; b^3 is $= 125{,}000$; and $\frac{b^3}{27}$ is $= \frac{125{,}000}{27} = 4629.629, 629, 629,$ &c. which is greater than 3600, or $\frac{cc}{4}$. Therefore this equation cannot be resolved by Cardan's second rule, (that was invented by *Tartaglia*,) but may be resolved by approximation by proceeding as follows.

Since b is $= 50$, $\sqrt{b}$ will be $= \sqrt{50} = 7.07$ &c. Therefore x (which is greater than $\sqrt{b}$, or $\sqrt{50}$) must be greater than 7.07 &c.

And,

And, ſince $\sqrt{b}$ is equal to 7.07 &c. it follows that $\frac{2\sqrt{b}}{\sqrt{3}}$, or $\frac{2000}{1732} \times \sqrt{b}$, will be equal to $\frac{2000}{1732} \times 7.07$, or $\frac{14140}{1732}$, or 8.16 &c. Therefore x (which is leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$,) will be leſs than 8.16 &c. We therefore may now conclude that x is greater than 7.07 &c. but leſs than 8.16 &c.

Further, when x is $= \sqrt{b}$, or $\sqrt{50}$, or 7.07 &c. $x^3 - bx$ is $(= b\sqrt{b} - b\sqrt{b}) = 0$; and when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, $x^3 - bx$ is $= \frac{2b\sqrt{b}}{3\sqrt{3}}$ $(= \frac{2 \times 50 \times \sqrt{50}}{3 \times 1.732 \text{ &c.}} = \frac{100 \times \sqrt{50}}{5.196 \text{ &c.}} = \frac{100 \times 7.07 \text{ &c.}}{5.196 \text{ &c.}} = \frac{707}{5.196 \text{ &c.}}) = 136$ &c. Therefore in the preſent equation $x^3 - 50x = 120$, in which $x^3 - 50x$ is equal to 120, which is not a great deal leſs than 136 &c. it ſeems reaſonable to ſuppoſe that x will approach nearer to it's higheſt limit 8.16 than to it's loweſt limit 7.07 &c. We will therefore ſuppoſe it to be nearly equal to 8, and will ſubſtitute 8 inſtead of it in the compound quantity $x^3 - 50x$, in order to diſcover whether it be greater or leſs than 8.

Art. XXXIII. If x is ſuppoſed to be $= 8$, we ſhall have $x^3 (= \overline{8}^3) = 512$, and $50x (= 50 \times 8) = 400$, and conſequently $x^3 - 50x (= 512 - 400) = 112$; which is ſomewhat leſs than the abſolute term 120. Therefore 8 muſt be ſomewhat leſs than the true value of x in the propoſed equation $x^3 - 50x = 120$. We will therefore take 8 for the firſt near value of the root of this equation,

tion, and the baſis of our further approximation to it's true value, and will ſubſtitute $8 + z$ inſtead of x in the terms of the ſaid equation, but with an omiſſion of all the terms of the cube and ſquare of z, agreeably to Mr. Raphſon's directions.

Now, if x is $= 8 + z$, we ſhall have $x^3 = \overline{8 + z}^3$

$$(= \overline{8}^3 + 3 \times \overline{8}^2 \times z + \&c$$
$$= 512 + 3 \times 64 \times z + \&c$$
$$= 512 + 192 \times z + \&c,$$

and $50x = 50 \times \overline{8 + z} (= 50 \times 8 + 50 \times z) = 400 + 50z$.

Therefore $x^3 - 50x$ will be $= 512 + 192z + \&c - 400 - 50z = 112 + 142z$.

But $x^3 - 50x$ is $= 120$.

Therefore $112 + 142z$ will alſo be $= 120$;

and conſequently $142z$ will be $(= 120 - 112) = 8$, and z will be $= \frac{8}{142} = 0.056$. Therefore $8 + z$, or x, will be nearly $= 8 + 0.056$, or 8.056.

Art. XXXIV. We muſt now ſubſtitute 8.056 inſtead of x in the compound quantity $x^3 - 50x$, which forms the left-hand ſide of the propoſed equation $x^3 - 50x = 120$, that we may diſcover whether the value of the ſaid compound quantity reſulting from that ſubſtitution will be greater or leſs than 120, or the abſolute term of the ſaid equation, and conſequently whether the ſaid number

8.056

8.056 is greater or leſs than the true value of x in the ſaid equation. This may be done as follows.

If x is $= 8.056$, we ſhall have x^3 $(= \overline{8.056}^3) =$ 522.827,439,616, and $50x$ $(= 50 \times 8.056) = 402.800$, and conſequently $x^3 - 50x$ $(= 522.827,439,616 - 402.800) = 120.027,439,616$, which is greater than 120. Therefore 8.056 is ſomewhat greater than the true value of x in the equation $x^3 - 50x = 120$.

Q. E. I.

We will therefore, in the next place, ſuppoſe x to be $= 8.056 - z$, and ſubſtitute the reſidual quantity $8.056 - z$ inſtead of x in the terms of the propoſed equation $x^3 - 50x = 120$.

Then, ſince x is $= 8.056 - z$, we ſhall have

$x^3 = \overline{8.056 - z}^3 \; (= \overline{8.056}^3 - 3 \times \overline{8.056}^2 \times z + \&c$
$= 522.827,439,616 - 3 \times 64.899,136 \times z + \&c)$
$= 522.827,439,616 - 194.697,408 \times z + \&c,$

and $50x = 50 \times \overline{8.056 - z} \; (= 50 \times 8.056 - 50 \times z)$
$= 402.800 - 50z,$

And conſequently

$x^3 - 50x = 522.827,439,616 - 194.697,408 \times z + \&c$

$-\overline{402.800 - 50z} = \left\{\begin{matrix} 522.827,439,616 - 194.697,408 \times z + \&c \\ -402.800 \qquad\qquad + 50.000,000 \times z \end{matrix}\right\}$

$= 120.027,439,616 - 144.697,408, \times z + \&c.$

But

But $x^3 - 50x$ is $= 120$.

Therefore 120.027,439,616 — 144.697,408 $\times z$ will alſo be $= 120$; and conſequently (adding 144.697,408 $\times z$ to both ſides,) 120.027,439,616 will be $= 120 +$ 144.697,408 $\times z$, and (ſubtracting 120 from both ſides,) 144.697,408 $\times z$ will be $=$ 0.027,439,616. Therefore z will be $= \frac{0.027,439,616}{144.697,408} =$ 0.000,189,63, and x, or 8.056 — z, will be $(= 8.056 -$ 0.000,189,63$) =$ 8.055,810,37; that is, the root of the propoſed equation $x^3 - 50x = 120$ will be $=$ 8.055,810,37.

Q. E. I.

Art. XXXV. The eight firſt figures of this value of x, (which has been obtained by only two proceſſes of approximation,) to wit, the figures 8.055,810,3, are exact; the more accurate value of x being 8.055,810,345, as will be found by carrying the approximation one ſtep further.

N. B. This equation $x^3 - 50x = 120$ expreſſes the relation between the diameter of a circle and three chords in it that lie contiguous to each other, and, together, take up the arch of a ſemicircle, and form a trapezium of which the diameter is the fourth ſide. For, if the three chords are called b, k, and t, and the diameter is called x, the relation between them will be expreſſed by the cubic equation $\left.\begin{matrix} x^3 - bb \\ - kk \\ - tt \end{matrix}\right\} \times x = 2bkt$; which, if the numbers 3, 4, and 5 are ſubſtituted inſtead of the letters b, k, and t, will become

$x^3 - 9$

$$\left.\begin{matrix} x^2 - 9 \\ -16 \\ -25 \end{matrix}\right\} \times x = 2 \times 3 \times 4 \times 5, \text{ or } x^3 - 50x (= 6 \times 20) = 120.$$

See Sir Iſaac Newton's *Arithmetica Univerſalis*, edition 2d, A. D. 1722, page 101.

EXAMPLE II.

Art. XXXVI. Let the equation to be reſolved be $x^3 - x = \frac{1}{3}$.

In this equation b is $= 1$, and c is $= \frac{1}{3}$; and conſequently $\frac{c}{2}$ is $\left(= \frac{1}{2} \times \frac{1}{3}\right) = \frac{1}{6}$, and $\frac{cc}{4}$ is $= \frac{1}{36}$; and $\frac{b}{3}$ is $= \frac{1}{3}$, and $\frac{b^3}{27}$ is $= \frac{1}{27}$, which is greater than $\frac{1}{36}$, or $\frac{cc}{4}$. Therefore this equation cannot be reſolved by Cardan's ſecond rule above-explained, but falls under the ſecond caſe of the general equation $x^3 - bx = c$, and muſt be reſolved by approximation. This may be done in the manner following.

Art. XXXVII. Since b is $= 1$, we ſhall have $\sqrt{b}$ alſo $= 1$; and conſequently x (which is greater than $\sqrt{b}$) muſt be greater than 1.

And,

And, ſince $\sqrt{b}$ is $= 1$, we ſhall have $\frac{2\sqrt{b}}{\sqrt{3}} = \frac{2 \times 1}{\sqrt{3}}$ $\left(= \frac{2.000 \times 1}{1.732 \text{ \&c}} = \frac{2000 \times 1}{1732} = \frac{2000}{1732}\right) = 1.154$ &c. Therefore x (which is always leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$) will be leſs than 1.154. We are therefore certain that x is greater than 1, but leſs than 1.154.

Further, when x is $= \sqrt{b}$, or $\sqrt{1}$, or 1, $x^3 - bx$ is $(= b\sqrt{b} - b\sqrt{b}) = 0$; and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, or 1.154, $x^3 - bx$ is $= \frac{2b\sqrt{b}}{3\sqrt{3}}$ $\left(= \frac{2 \times 1 \times 1}{3 \times 1.732 \text{ \&c}} = \frac{2}{5.196} = 0.384\right)$. Therefore in the preſent caſe, in which $x^3 - bx$ is $= \frac{1}{3}$, or 0.333, &c, which is not very much leſs than 0.384, it ſeems reaſonable to ſuppoſe that x will be nearer to it's higher limit 1.154 than to it's lower limit 1, and conſequently that it will be greater than half the ſum of thoſe limits, or than $\frac{1 + 1.154}{2}$, or $\frac{2.154}{2}$, or 1.077. We will therefore ſuppoſe it to be $= 1.1$, and will ſubſtitute 1.1 in it's ſtead in the compound quantity $x^3 - x$, in order to diſcover whether the value of the ſaid compound quantity reſulting from that ſubſtitution will be greater or leſs than $\frac{1}{3}$, or 0.333, &c, or the abſolute term of the propoſed equation $x^3 - x = \frac{1}{3}$, or $x^3 - x = 0.333$, &c, and conſequently whether 1.1 is greater or leſs than the true value of x in the ſaid equation.

Now, if x be $= 1.1$, we ſhall have $x^3 = 1.331$, and conſequently

conſequently $x^3 - x$ (= 1.331 — 1.1) = 0.231; which is leſs than 0.333, &c, or the abſolute term of the equation $x^3 - x = 0.333$, &c. Therefore 1.1 muſt be leſs than the true value of x in the ſaid equation.

Art. XXXVIII. Let us therefore ſuppoſe x to be = 1.1 + z, and let this binomial quantity 1.1 + z be ſubſtituted inſtead of x in the propoſed equation $x^3 - x = 0.333$, &c, but with an omiſſion of all the terms that involve either the cube or ſquare of z, agreeably to Mr. Raphſon's directions.

Then, ſince x is = 1.1 + z, we ſhall have

$$x^3 = \overline{1.1+z}^3 \; (= \overline{1.1}^3 + 3 \times \overline{1.1}^2 \times z + \&c =$$
$$1.331 + 3 \times 1.21 \times z + \&c) =$$
$$1.331 + 3.63 \times z + \&c,$$

and conſequently $x^3 - x = 1.331 + 3.63 \times z -$ $\overline{1.1+z}$ (= 1.331 + 3.63 × z — 1.1 — z) = 0.231 + 2.63 × z.

But $x^3 - x$ is = 0.333, &c.
1.1

Therefore 0.231 + 2.63 × z will alſo be = 0.333, &c; and conſequently (ſubtracting 0.231 from both ſides) 2.63 × z will be = 0.102,333, &c, and z will be = $\frac{0.102,333, \&c.}{2.63}$ = 0.038. Therefore x, or 1.1 + z, will be = 1.1 + 0.038 = 1.138; that is, the ſecond near value of the root of the propoſed equation $x^3 - x = \frac{1}{3}$, or $x^3 - x$ = 0.333,333, &c, will be 1.138.

Art.

Art. XXXIX. This value of x must now be ſubſtituted inſtead of x in the compound quantity $x^3 - x$, in order to diſcover whether the value of the ſaid compound quantity reſulting from ſuch ſubſtitution will be greater or leſs than 0.333,333, &c, or the abſolute term of the equation $x^3 - x = 0.333,333$, &c, and conſequently whether 1.138 is greater or leſs than the true value of x in the ſaid equation. This ſubſtitution will be as follows.

Now, if x is ſuppoſed to be $= 1.138$, we ſhall have $x^3 (= \overline{1.138}^3) = 1.473,760,072$, and conſequently $x^3 - x (= 1.473,760,072 - 1.138) = 0.335,760,072$; which is greater than 0.333,333, &c, or the abſolute term of the equation $x^3 - x = 0.333,333$, &c. Therefore 1.138 is greater than the true value of x in the ſaid equation.

Q. E. I.

Art. XL. We will therefore, in the next place, ſuppoſe x to be $= 1.138 - z$, and ſubſtitute this reſidual quantity inſtead of it in the equation $x^3 - x = 0.333,333$, &c.

Since x is $= 1.138 - z$, we ſhall have

$$x^3 = \overline{1.138 - z}^3 (= \overline{1.138}^3 - 3 \times \overline{1.138}^2 \times z + \&c$$
$$= 1.473,760,072 - 3 \times 1.295,044 \times z + \&c)$$
$$= 1.473,760,072 - 3.885,132 \times z + \&c, \text{ and conſequently}$$
$$x^3 - x = 1.473,760,072 - 3.885,132 \times z + \&c$$
$$- \overline{1.138 - z} = \left\{ \begin{array}{l} 1.473,760,072 - 3.885,132 \times z + \&c \\ -1.138,000,000 + 1.000,000 \times z \end{array} \right\}$$
$$= 0.335,760,072 - 2.885,132 \times z + \&c.$$

But

But $x^3 - x$ is $= 0.333,333,333$, &c.

Therefore $0.335,760,072 - 2.885,132 \times z$ will alſo be $= 0.333,333,333$, &c; and conſequently (adding $2.885,132 \times z$ to both ſides,) $0.335,760,072$ will be $= 0.333,333,333$, &c $+ 2.885,132 \times z$; and (ſubtracting $0.333,333,333$, &c from both ſides,) $2.885,132 \times z$ will be $= 0.002,426,739$. Therefore z will be $= \frac{0.002,426,739}{2.885,132} = 0.000,841,11$; and conſequently x, or $1.138 - z$, will be $= 1.138 - 0.000,841,11 = 1.137,158,89$; that is, the third near value of the root of the propoſed equation $x^3 - x = 0.333,333,333$, &c, or $x^3 - x = \frac{1}{3}$, will be $1.137,158,89$.

Q. E. I.

This value of the root of the equation $x^3 - x = \frac{1}{3}$ is exact in the firſt ſeven figures, $1.137,158$; the more accurate value of it being $1.137,158,164$, as will be found by carrying the approximation one ſtep further.

EXAMPLE III.

Art. XLI. Let the equation to be reſolved be $x^3 - 3x = 4$.

Here b is $= 3$, and c is $= 4$; and conſequently $\frac{c}{z}$ is

is $= 2$, and $\frac{cc}{4}$ is $= 4$, and $\frac{b^3}{27}$ is $= \frac{125}{27} = 4.629,629,629,$ 629, &c, which is greater than 4, or $\frac{cc}{4}$. Therefore this equation cannot be reſolved by Cardan's ſecond rule above-explained, but comes under the ſecond caſe of the general equation $x^3 - bx = c$, and muſt be reſolved by approximation.

Art. XLII. Now, ſince b is $= 5$, we ſhall have $\sqrt{b} = \sqrt{5} = 2.236$, &c, and $\frac{2\sqrt{b}}{\sqrt{3}} = \frac{2 \times 2.236 \text{ &c}}{1\,732, \text{ &c}} = \frac{4.47^{\circ} \text{ &c}}{1.732 \text{ &c}} = 2.581$ &c. Therefore x will be greater than 2.236 &c, but leſs than 2.581, &c.

Further, when x is $= \sqrt{b}$, or $\sqrt{5}$, $x^3 - bx$, or $x^3 - 5x$, will be $(= b\sqrt{b} - b\sqrt{b}$, or $5\sqrt{5} - 5\sqrt{5}) = 0$; and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, or 2.581, &c. $x^3 - bx$, or $x^3 - 5x$, will be $= \frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{2 \times 5 \times \sqrt{5}}{3 \times \sqrt{3}}$, (or $\frac{10 \times \sqrt{5}}{3 \times \sqrt{3}}$, or $\frac{10 \times 2.236, \text{ &c}}{3 \times 1.732}$, or $\frac{22\,36 \text{ &c}}{5.196}$,) or 4.303, &c. But in the preſent caſe $x^3 - 5x$ is ~~equal to 4,~~ which is not much leſs than 4.303, &c, or it's higher limit. Therefore it ſeems reaſonable to ſuppoſe that x will not be much leſs than it's higher limit 2.581. We will therefore ſuppoſe it to be $= 2.5$, and will ſubſtitute 2.5 inſtead of it in the compound quantity $x^3 - 5x$, in order to diſcover whether the value of the ſaid compound quantity reſulting from ſuch ſubſtitution will be greater, or leſs, than 4, or the abſolute term of the propoſed equation $x^3 - 5x = 4$, and conſequently

quently whether 2.5 will be greater, or leſs, than the true value of x in the ſaid equation.

Now, if we ſuppoſe x to be $= 2.5$, we ſhall have $x^3 = 15.625$, and $5x\ (= 5 \times 2.5) = 12.5$, and conſequently $x^3 - 5x\ (= 15.625 - 12.500) = 3.125$; which is leſs than 4, or the abſolute term of the equation $x^3 - 5x = 4$. Therefore 2.5 will be leſs than the value of x in that equation.

Q. E. I.

Art. XLIII. We will therefore, in the next place, ſuppoſe x to be $= 2.5 + z$, and ſubſtitute this binomial quantity inſtead of x in the equation $x^3 - 5x = 4$.

Then, ſince x is $= 2.5 + z$, we ſhall have

$$x^3 = \overline{2.5 + z}^3\ (= \overline{2.5}^3 + 3 \times \overline{2.5}^2 \times z + \&c$$
$$= 15.625 + 3 \times 6.25 \times z + \&c)$$
$$= 15.625 + 18.75 \times z + \&c,$$

and $5x = 5 \times \overline{2.5+z}\ (= 5 \times 2.5 + 5 \times z) = 12.5 + 5z$,

and conſequently $x^3 - 5x = \left\{\begin{array}{r} 15.625 + 18.75 \times z \\ -12.500 - 5.00 \times z \end{array}\right\}$

$$= 3.125 + 13.75 \times z.$$

But $x^3 - 5x$ is $= 4$.

Therefore $3.125 + 13.75 \times z$ will alſo be $= 4$; and conſequently $13.75 \times z$ will be $(= 4 - 3.125) = 0.875$, and z will be $= \frac{0.875}{13.75} = 0.063$. Therefore x, or $2.5 + z$,

 will

will be = 2.5 + 0.063 = 2.563; or the ſecond near value of the root of the propoſed equation $x^3 - 5x = 4$ will be 2.563.

Q. E. I.

Art. XLIV. We muſt next proceed to ſubſtitute 2.563 inſtead of x in the compound quantity $x^3 - 5x$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than 4, or the abſolute term of the equation $x^3 - 5x = 4$, and conſequently whether 2.563 is greater, or leſs, than the true value of x in that equation.

Now, if x be = 2.563, we ſhall have x^3 = 16.836,267,547, and $5x$ = 12.815, and conſequently $x^3 - 5x$ (= 16.836,267,547 — 12 815) = 4.021,267,547; which is greater than 4. Therefore 2.563 is greater than the true value of x in the equation $x^3 - 5x = 4$.

Q. E. I.

Art. XLV. We will therefore, in the next place, ſuppoſe x to be = 2.563 — z, and ſubſtitute this reſidual quantity inſtead of x in the equation $x^3 - 5x = 4$, and reſolve the equation thence reſulting as if it were a mere ſimple equation.

Now, if x be ſuppoſed to be = 2.563 — z, we ſhall have

$x^3 = \overline{2.563-z}^3$ (= $\overline{2.563}^3 - 3 \times \overline{2.563}^2 \times z$ + &c

= 16.836,267,547 — 3 × 6.568,969 × z + &c)

= 16.836,267,547 — 19.706,907 × z + &c,

and $5x = 5 \times \overline{2.563-z}$ (= 5 × 2.563 — 5z) = 12.815 — 5z,

and

and consequently $x^3 - 5x$ (= 16.836,267,547 − 19.706,907 × z + &c − 12.815 − $5z$ = 16.836,267,547 − 19.706,907 × z − 12.815 + $5z$ = 4.021,267,547 − 14.706,907 × z.

But $x^3 - 5x$ is = 4.

Therefore 4.021,267,547 − 14.706,907 × z will also be = 4; and consequently (adding 14.706,907 × z to both sides,) 4.021,267,547 will be = 4 + 14.706,907 × z, and (subtracting 4 from both sides,) 14.706,907 × z will be = 0.021,267,547. And consequently z will be = $\frac{0.021,267,547}{14.706,907}$ = 0.001,446. Therefore x, or 2.563 − z, will be = 2.563 − 0.001,446 = 2.561,554; that is, the third near value of the root of the proposed equation $x^3 - 5x = 4$ will be 2.561,554.

Q. E. I.

Art. XLVI. Now let 2.561,554 be substituted instead of x in the compound quantity $x^3 - 5x$, in order to discover whether the result will be greater, or less, than 4, or the absolute term of the proposed equation $x^3 - 5x = 4$, and consequently whether 2.561,554 is greater, or less, than the true value of x in that equation.

If x is = 2.561,554, we shall have xx = 6.561,558,894,916, and x^3 = 16.807,787,433,507,659,464, and $5x$ (= 5 × 2.561,554) = 12.807,770, and consequently $x^3 - 5x$ (= 16.807,787,433,507,659,464 − 12.807,770) = 4.000,017,433,507,659,464; which is a little greater than 4, or the absolute term of the equation $x^3 - 5x = 4$.

Therefore 2.561,554 is a little greater than the true value of x in that equation.

Q. E. I.

Art. XLVII. We will therefore now ſuppoſe x to be $= 2.561,554 - z$, and will ſubſtitute that reſidual quantity inſtead of x in the equation $x^3 - 5x = 4$, and then reſolve the equation reſulting from ſuch ſubſtitution as if it were a mere ſimple equation. This may be done as follows:

Since x is $= 2.561,554 - z$, we ſhall have

$$x^3 = \overline{2.561,554 - z}^3 \; (= \overline{2.561,554}^3$$
$$- 3 \times \overline{2.561,554}^2 \times z + \&c =$$
$$\overline{2.561,554}^3 - 3 \times 6.561,558,894,916 \times z + \&c$$
$$= \overline{2.561,554}^3 - 19.684,676,684,748 \times z + \&c)$$
$$= 16.807,787,433,507,659,464 - 19.684,676,684,748 \times z + \&c,$$

and $5x = 5 \times \overline{2.561,554 - z} = 5 \times 2.561,554$
$$- 5z = 12.807,770 - 5z.$$

Therefore $x^3 - 5x$ will be $=$

$$16.807,787,433,507,659,464 - 19.684,676,684,748 \times z + \&c$$
$$- \overline{12.807,770 - 5z} =$$
$$\{16.807,787,433,507,659,464 - 19.684,676,684,748 \times z + \&c\}$$
$$\{-12.807,770,000,000,000,000 + 5.000,000,000,000 \times z + \&c\}$$
$$= 4.000,017,433,507,659,464 - 14.684,676,684,748 \times z + \&c.$$

But

But $x^3 - 5x$ is $= 4$.

Therefore 4.000,017,433,507,659,464 $-$ 14,684,676,684,748 $\times z$ + &c will be $= 4$; and conſequently (adding 14.684,676,684,748 $\times z$ to both ſides,) 4.000,017,433,507,659,464 will be $= 4 +$ 14.684,676,684,748 $\times z$; and (ſubtracting 4 from both ſides,) 14.684,676,684,748 $\times z$ will be $=$ 0.000,017,433,507,659,464, and z will be $= \frac{0.000,017,433,507,659,464}{14.684,676,684,748} =$ 0.000,001,187,190.

Therefore x, or 2.561,554 $- z$, will be ($=$ 2.561,554 $-$ 0.000,001,187,190,) $=$ 2.561,552,812,810; that is, the fourth near value of the root of the propoſed equation $x^3 - 5x = 4$ will be $=$ 2.561,552,812,810.

Q. E. I.

Art. XLVIII. Of this number 2.561,552,812,810 the firſt eleven figures 2.561,552,812,8 are certainly exact. For the accurate value of x in this equation is $\frac{1+\sqrt{17}}{2}$, or $\frac{1 + 4.123,105,625,6 \&c}{2}$, or $\frac{5.123,105,625,6 \&c}{2}$, or 2.561,552,812,8 &c. For, if we ſubſtitute $\frac{1+\sqrt{17}}{2}$ inſtead of x in the compound quantity $x^3 - 5x$, we ſhall find, that the ſaid quantity will be equal to 4, or the abſolute term of the equation $x^3 - 5x = 4$. For, if x is $= \frac{1+\sqrt{17}}{2}$, we

shall have $x^3 = \left(\frac{1+\sqrt{17}}{2}\right)^3$ $\left(= \frac{1+\sqrt{17}^3}{2^3} = \frac{1+\sqrt{17}^3}{8} = \right.$

$\left.\frac{1^3 + 3 \times 1^2 \times \sqrt{17} + 3 \times 1 \times 17 + 17 \times \sqrt{17}}{8}\right.$

$= \frac{1 + 3 \times 1 \times \sqrt{17} + 3 \times 17 + 17 \times \sqrt{17}}{8} =$

$\frac{1 + 3 \times \sqrt{17} + 51 + 17 \times \sqrt{17}}{8} = \frac{52 + 20 \times \sqrt{17}}{8} =$

$\left.\frac{4 \times 13 + 4 \times 5 \times \sqrt{17}}{4 \times 2}\right) = \frac{13 + 5 \times \sqrt{17}}{2}$, and $5x =$

$5 \times \frac{1+\sqrt{17}}{2}$ $\left(= \frac{5 \times 1 + \sqrt{17}}{2} = \frac{5 \times 1 + 5 \times \sqrt{17}}{2}\right)$

$= \frac{5 + 5 \times \sqrt{17}}{2}$; and consequently $x^3 - 5x = \frac{13+5 \times \sqrt{17}}{2}$

$- \frac{5+5 \times \sqrt{17}}{2}$ $\left(= \frac{13 + 5 \times \sqrt{17} - 5 - 5 \times \sqrt{17}}{2} = \right.$

$\left.\frac{13 - 5}{2} = \frac{8}{2}\right) = 4$, or the absolute term of the equation

$x^3 - 5x = 4$. Therefore $\frac{1+\sqrt{17}}{2}$ must be $= x$, or the root of the said equation.

Q. E. D.

OBSERVATIONS ON THE FOREGOING EXAMPLE.

Art. XLIX. By this example it appears that, after we have obtained the first near value of x by a conjecture founded on the knowledge that it must be of an intermediate magnitude between the known quantities $\sqrt{b}$ and $\frac{2\sqrt{b}}{\sqrt{3}}$ or $\frac{2000}{1732} \times \sqrt{b}$, and on a comparison of the absolute term c, (which in this case is 4,) with $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or the greatest

eſt poſſible magnitude of the abſolute term of an equation of this kind, (which in the equation $x^3 - 5x = 4$ is 4.303 &c,) and have made the firſt near value of x ſo obtained, to wit, the number 2.5, the baſis of an approximation to it's more accurate value in the method preſcribed by Mr. Raphſon, three proceſſes of this method have been ſufficient to give us the root of the propoſed equation $x^3 - 5x = 4$ exact to eleven places of figures. And it will be found by thoſe who will take the trouble of trying it, that the ſame number of proceſſes of approximation will give us the root of the equation to the ſame degree of exactneſs in any other equation of the ſame kind, or that ſhall come under the ſecond caſe of the general equation $x^3 - bx = c$; becauſe the proximity of the two limits $\sqrt{b}$ and $\frac{2\sqrt{b}}{\sqrt{3}}$, or $\frac{2000}{1732} \times \sqrt{b}$, between which the value of x muſt always lie, will always enable us to obtain a very convenient firſt near value of it to begin our approximation. And the number of new figures that are exact in the value of x that is obtained by each new proceſs of approximation, will uſually be equal, wanting one figure, to that of the figures that were exact in the next preceeding near value of x upon which the ſaid proceſs of approximation was grounded. Thus we have ſeen that, when we had obtained 2.561,554 for a near value of x, in which number the firſt ſix figures 2.561,55 are exact, the next near value of x, or that which was obtained by the next proceſs of approximation, was 2.561,552, 812,810, in which value five additional figures, to wit, the figures 28128, which follow the firſt ſix figures 2.561,55, are alſo exact. And the ſame thing will be found to take place in all other inſtances.

EXAMPLE IV.

Art. L. LET it be required to find the root of the cubick equation $x^3 - 63x = 162$.

Here b is $= 63$, and c is $= 162$; and conſequently $\frac{c}{2}$ is $= 81$, and $\frac{cc}{4}$ is $(= \overline{81}^2) = 6561$; and $\frac{b}{3}$ is $\left(= \frac{63}{3}\right) = 21$, and $\frac{b^3}{27}$, or $\overline{\frac{b}{3}}^3$, is $(= \overline{21}^3) = 9261$, which is greater than 6561, or $\frac{cc}{4}$. Therefore this equation cannot be reſolved by Cardan's ſecond rule above-explained, but falls under the ſecond caſe of the general equation $x^3 - bx = c$, and muſt be reſolved by approximation. This may be done in the manner following:

Art. LI. Since b is $= 63$, we ſhall have $\sqrt{b} = \sqrt{63} = 7.937$, &c, and $\frac{2\sqrt{b}}{\sqrt{3}} = \frac{2 \times 7.937, \&c}{\sqrt{3}}$ $\left(= \frac{2 \times 7.937 \&c}{1.732 \&c.} = \frac{15.874, \&c}{1.732, \&c}\right) = 9.165$, &c. Therefore x (which is always greater than $\sqrt{b}$, but leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$,) will be greater than 7.937, &c, but leſs than 9.165, &c.

Further, when x is $= \sqrt{b}$, or $\sqrt{63}$, (which is it's lower limit,) the compound quantity $x^3 - bx$, or $x^3 - 63x$, will be $(= b\sqrt{b} - b\sqrt{b}$, or $63 \times \sqrt{63} - 63 \times \sqrt{63},) = 0$; and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, or 9.165 &c, it's higher limit, the compound

compound quantity $x^3 - bx$, or $x^3 - 63x$, will be $= \frac{2b \times \sqrt{b}}{3\sqrt{3}} = \frac{2 \times 63 \times \sqrt{63}}{3\sqrt{3}} \left(= \frac{2 \times 63 \times 7.937 \text{ \&c}}{3 \times 1.732, \text{ \&c}} = \frac{2 \times 21 \times 7.937 \text{ \&c}}{1.732, \text{ \&c}} = \frac{42 \times 7.937 \text{ \&c}}{1.732, \text{ \&c}} = \frac{333.354, \text{ \&c}}{1.732, \text{ \&c}} \right) =$ 192, &c. But in the preſent caſe $x^3 - 63x$ is equal to 162, which is more than $\frac{4}{5}$ of 192, &c. Therefore we may reaſonably conjecture that the value of x in the equation $x^3 - 63x = 162$ will approach nearer to it's higher limit 9.165 &c than to its lower limit 7.937, &c, or will be greater than half the ſum of the ſaid limits, (or than $\frac{7.937 \text{ \&c} + 9.165, \text{ \&c}}{2}$, or $\frac{17.102 \text{ \&c}}{2}$,) or 8.551. We will therefore ſuppoſe it to be = 8.7, and will proceed to ſubſtitute 8.7 inſtead of x in the compound quantity $x^3 - 63x$, in order to diſcover whether the reſult of ſuch ſubſtitution will be greater, or leſs, than 162, or the abſolute term of the propoſed equation $x^3 - 63x = 162$, and conſequently whether 8.7 will be greater, or leſs, than the true value of x in the ſaid equation.

Now, if x is $= 8.7$, we ſhall have $x^3 (= \overline{8.7}^3) = 658.503$, and $63x (= 63 \times 8.7) = 548.1$. Therefore $x^3 - 63x$ will be $(= 658.503 - 548.1) = 110.403$; which is leſs than 162, or the abſolute term of the equation $x^3 - 63x = 162$. Therefore 8.7 muſt be leſs than the true value of x in that equation.

Q. E. I.

Art. LII. We will therefore put $x = 8.7 + z$, and ſubſtitute the binomial quantity $8.7 + z$ inſtead of x in

the equation $x^3 - 63x = 162$, and then reſolve the new equation reſulting from ſuch ſubſtitution as if it were a mere ſimple equation. This may be done as follows:

Since x is $= 8.7 + z$, we ſhall have

$$x^3 = \overline{8.7+z}^3 (= \overline{8.7}^3 + 3 \times \overline{8.7}^2 \times z + \&c$$
$$= \overline{8.7}^3 + 3 \times 75.69 \times z + \&c$$
$$= \overline{8.7}^3 + 227.07 \times z + \&c)$$
$$= 658.503 + 227.07 \times z + \&c,$$

and $63x = 63 \times \overline{8.7 + z} (= 63 \times 8.7 + 63 \times z)$
$$= 548.1 + 63 \times z.$$

Therefore $x^3 - 63x$ will be $(= 658.503 + 227.07 \times z - 548.1 - 63 \times z) = 110.403 + 164.07 \times z$.

But $x^3 - 63x$ is $= 162$.

Therefore $110.403 + 164.07 \times z$ will alſo be $= 162$; and conſequently $164.07 \times z$ will be $(= 162 - 110.403) = 51.597$; and z will be $= \frac{51.597}{164.07} = 0.31$.

Therefore x or $8.7 + z$ will be $= 8.7 + 0.31 = 9.01$; that is, the ſecond near value of the root of the propoſed equation $x^3 - 63x = 162$ will be 9.01.

Art. LIII. We will now ſubſtitute 9.01 in the compound quantity $x^3 - 63x$, in order to diſcover whether the reſult will be greater, or leſs, than 162, or the abſolute term of the equation $x^3 - 63x = 162$, and conſequently

quently whether 9.01 is greater, or leſs, than the true value of x in that equation.

Now, if x is $= 9.01$, we ſhall have x^3 $(= \overline{9.01}^3) =$ 731.432,701, and $63x$ $(= 63 \times 9.01) =$ 567.63, and conſequently $x^3 - 63x$ $(= 731.432,701 - 567.63) =$ 163.802,701; which is greater than 162, or the abſolute term of the propoſed equation $x^3 - 63x = 162$. Therefore 9.01 muſt be greater than the true value of x in the ſaid equation.

Q. E. I.

Art. LIV. Since 9.01 is greater than the true value of x in the equation $x^3 - 63x = 162$, it ſeems reaſonable to conjecture that the whole number 9 itſelf may be exactly equal to it. And ſo, upon trial, we ſhall find it to be. For, if x is $= 9$, we ſhall have $x^3 = 729$, and $63x$ $(= 63 \times 9)$ $= 567$, and conſequently $x^3 - 63x$ $(= 729 - 567) =$ 162. Therefore x, or the root of the equation $x^3 - 63x$ $= 162$, is exactly equal to the whole number 9.

Q. E. I.

EXAMPLE V.

Art. LV. Let the equation that is to be reſolved be $x^3 - 15x = 4$.

Here b is $= 15$, and c is $= 4$; and conſequently $\frac{c}{2}$ is

is = 2, and $\frac{cc}{4}$ is $(= \overline{2}^2) = 4$, and $\frac{b}{3}$ is = 5, and $\frac{b^3}{27}$ is $(= \overline{5}^3) = 125$, which is greater than 4, or $\frac{cc}{4}$. Therefore this equation cannot be reſolved by Cardan's ſecond rule above-explained, but falls under the ſecond caſe of the general equation $x^3 - bx = c$, and muſt be reſolved by approximation; which may be done in the manner following:

Since b is = 15, we ſhall have $\sqrt{b}$ $(= \sqrt{15}) = 3.872$, &c, and $\frac{2\sqrt{b}}{\sqrt{3}} = \frac{2 \times 3.872, \&c}{\sqrt{3}}$ $(= \frac{2 \times 3.872, \&c}{1.732, \&c} = \frac{7.744, \&c}{1.732})$ = 4.471, &c. Therefore x (which is always greater than $\sqrt{b}$, but leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$,) will be greater than 3.872, &c, but leſs than 4.471, &c.

Art, LVI. Further, when x is $= \sqrt{b}$, or $\sqrt{15}$, the compound quantity $x^3 - bx$, or $x^3 - 15x$, is $(= b\sqrt{b} - b\sqrt{b}$, or $15\sqrt{15} - 15\sqrt{15}) = 0$; and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, or $\frac{2\sqrt{15}}{\sqrt{3}}$, or 4.471, &c, the ſame compound quantity will be $= \frac{2b\sqrt{b}}{3\sqrt{3}} = \frac{2 \times 15 \times \sqrt{15}}{3\sqrt{3}}$ $(= \frac{2 \times 5 \times \sqrt{15}}{\sqrt{3}}$ $= \frac{2 \times 5 \times 3.872 \&c.}{1.732, \&c} = \frac{10 \times 3.872, \&c}{1.732, \&c} = \frac{38.720 \&c}{1.732})$ = 22.355, &c; which is much greater than the abſolute term 4, or the value of the ſaid compound quantity in the propoſed equation $x^3 - 15x = 4$. Therefore it ſeems reaſonable to ſuppoſe that the value of x in that equation will differ leſs from it's lower limit, 3.872, &c, than from

from it's higher limit 4.471, &c, and consequently will be less than half the sum of the said limits, (or than $\frac{3.872 \text{ &c} + 4.471 \text{ &c}}{2}$, or $\frac{8.343, \text{ &c}}{2}$,) or than 4.171, &c. We will therefore suppose it to be equal to the whole number 4, and will proceed to substitute 4 instead of x in the compound quantity $x^3 - 15x$, in order to discover whether the result of such substitution will be equal to, or greater than, or less than, 4, or the absolute term of the proposed equation $x^3 - 15x = 4$, and consequently whether the said number 4 will be equal to, greater than, or less than, the true value of x in the said equation.

Now, if x be $= 4$, we shall have x^3 $(= \overline{4}^3) = 64$, and $15x$ $(= 15 \times 4) = 60$, and consequently $x^3 - 15x$ $(= 64 - 60) = 4$. Therefore the root of the proposed equation $x^3 - 15x = 4$ will be the whole number 4.

Q. E. I.

EXAMPLE VI.

Art. LVII. LET the equation that is to be resolved be $x^3 - 90x = 98$.

Here b is $= 90$, and c is $= 98$; and consequently $\frac{c}{2}$ is $= 49$, and $\frac{cc}{4}$ is $(= \overline{49}^2) = 2401$, and $\frac{b}{3}$ is $(= \frac{90}{3})$ $= 30$,

$= 30$, and $\frac{b^3}{27}$ is $(= \overline{3c}^3) = 27{,}000$, which is greater than 2401, or $\frac{cc}{4}$. Therefore this equation cannot be reſolved by Cardan's ſecond rule above-explained, but falls under the ſecond caſe of the general equation $x^3 - bx = c$, and muſt be reſolved by approximation; which may be done as follows:

Since b is in this caſe $= 90$, we ſhall have $\sqrt{b}\ (= \sqrt{90}) = 9.486$, &c, and $\frac{2\sqrt{b}}{\sqrt{3}} = \frac{2 \times 9.486, \&c}{1.732, \&c}\ \left(= \frac{18.972, \&c}{1.732, \&c}\right) = 10.953$, &c. Therefore x (which is always greater than $\sqrt{b}$, but leſs than $\frac{2\sqrt{b}}{\sqrt{3}}$) will be greater than 9.486, &c, but leſs than 10.953, &c.

Art. LVIII. Further, when x is $= \sqrt{b}$, or $\sqrt{90}$, the compound quantity $x^3 - bx$, or $x^3 - 90x$, will be $(= b\sqrt{b} - b\sqrt{b}$, or $90\sqrt{90} - 90\sqrt{90}) = 0$; and, when x is $= \frac{2\sqrt{b}}{\sqrt{3}}$, or $\frac{2\sqrt{90}}{\sqrt{3}}$, or 10.953, &c, the compound quantity $x^3 - bx$, or $x^3 - 90x$, will be $= \frac{2b\sqrt{b}}{3\sqrt{3}} = \frac{2 \times 90 \times \sqrt{90}}{3\sqrt{3}} = \frac{2 \times 30 \times \sqrt{90}}{\sqrt{3}} = \frac{60 \times 9.486, \&c.}{1.732, \&c} = \frac{560.160}{1.732, \&c} = 328.614$ &c. But in the equation $x^3 - 90x = 98$ the compound quantity $x^3 - 90x$, being equal to 98, differs much leſs from 0, or it's value when x is equal to it's lower limit $\sqrt{90}$, or 9.486, &c, than from 328.614, which is it's value when x is equal to it's higher limit

limit 10.593, &c. Therefore it ſeems reaſonable to ſuppoſe that the true value of x in this equation will differ leſs from it's lower limit 9.486, &c, than from it's higher limit 10.953, &c, and conſequently will be leſs than half the ſum of it's ſaid limits, or than $\frac{9.486, \&c + 10.953, \&c}{2}$, (or $\frac{20.439, \&c}{2}$,) or 10.219, &c. We will therefore ſuppoſe it to be $= 10$, and will proceed to ſubſtitute 10 inſtead of x in the compound quantity $x^3 - 90x$, (which forms the left-hand ſide of the propoſed equation $x^3 - 90x = 98$,) in order to diſcover whether the reſult of ſuch ſubſtitution will be equal to, or greater than, or leſs than, 98, or the abſolute term of the ſaid equation, and conſequently whether the ſaid number 10 will be equal to, or greater than, or leſs than, the true value of x in the ſaid equation.

Now, if x be $= 10$, we ſhall have $x^3 (= \overline{10}^3) = 1000$, and $90x (= 90 \times 10) = 900$, and conſequently $x^3 - 90x (= 1000 - 900) = 100$, which is greater than 98. Therefore 10 will be greater than the true value of x in the equation $x^3 - 90x = 98$.

Q. E. I.

Art. LIX. We will therefore now ſuppoſe x to be $= 10 - z$, and will ſubſtitute this reſidual quantity $10 - z$ inſtead of x in the propoſed equation $x^3 - 90x = 98$, and will then reſolve the new equation ariſing from ſuch ſubſtitution in the ſame manner as a mere ſimple equation, agreeably to the directions of Mr. Raphſon.

Since

Since x is $= 10 - z$, we ſhall have

$$x^3 = \overline{10-z}^3 \left(= \overline{10}^3 - 3 \times \overline{10}^2 \times z + \&c = 1000 - 3 \times 100 \times z + \&c\right) = 1000 - 300z + \&c,$$

and $90x = 90 \times \overline{10 - z}\ (= 90 \times 10 - 90 \times z) = 900 - 90z$,

and conſequently $x^3 - 90x = 1000 - 300z + \&c$

$$-\overline{900 - 90z} = \left\{\begin{matrix} 1000 - 300z + \&c \\ -900 + 90z \end{matrix}\right\} =$$

$$100 - 210z + \&c.$$

But $x^3 - 90x$ is $= 98$.

Therefore $100 - 210z + \&c$ will alſo be $= 98$; and conſequently (adding $210z$ to both ſides), 100 will be $= 98 + 210z$, and (ſubtracting 98 from both ſides,) $210z$ will be $= 2$, and z will be $= \frac{2}{210} = \frac{1}{105} = 0.009,523$.

Therefore x, or $10 - z$, will be $= 10 - 0.009,523 = 9.990,477$, or the root of the propoſed equation $x^3 - 90x = 98$ will be nearly equal to 9.990,477.

Q. E. I.

Art. LX. We will now ſubſtitute this number, 9.990,477, inſtead of x, in the compound quantity $x^3 - 90x$, in order to diſcover whether the reſult will be greater, or leſs, than 98, or the abſolute term of the equation $x^3 - 90x = 98$, and conſequently whether the ſaid number 9.990,477 is greater, or leſs, than the true value of x in that equation.

Now,

Now, if x is $= 9.990,477$, we ſhall have x^3 $(= \overline{9.990,477}^3) = 997.145,819,762,252,661,333$, and $90x$ $(= 90 \times 9.990,477) = 899.142,930$, and conſequently

$$x^3 - 90x = \left\{\begin{array}{l} 997.145,819,762,252,661,333 \\ -899.142,930 \end{array}\right\}$$

$= 98.002,889,762,252,661,333$; which is a little greater than 98, or the abſolute term of the propoſed equation $x^3 - 90x = 98$. Therefore the number $9.990,477$ is ſomewhat greater than the true value of x in that equation.

Q. E. I.

Art. LXI. In order therefore to obtain a more accurate value of the root of this equation, let us ſuppoſe x to be $= 9.990,477 - z$, and let us ſubſtitute this reſidual quantity inſtead of x in the terms of the ſaid equation, and then reſolve the new equation reſulting from ſuch ſubſtitution as if it were a ſimple equation. This may be done as follows:

Since x is $= 9.990,477 - z$, we ſhall have

$$x^3 = \overline{9.990,477 - z}^3 \ (= 9.990,477^3$$
$$- 3 \times \overline{9.990,477}^2 \times z + \&c =$$
$$\overline{9.990,477}^3 - 3 \times 99.809,630,687,529, \times z + \&c$$
$$= \overline{9.990,477}^3 - 299.428,892,062,587 \times z + \&c)$$
$$= 997.145,819,762,252,661,333$$
$$- 299.428,892,062,587 \times z + \&c,$$

and $90x = 90 \times \overline{9.990,477 - z}$
$(= 90 \times 9.990,477 - 90 \times z)$
$= 899.142,930 - 90z,$

and consequently $x^3 - 90x =$

$997.145,819,762,252,661,333 - 299.428,892,062,587 \times x$

$- 899.142,930 - 90z =$

$$\left\{\begin{array}{l} 997.145,819,762,252,661,333 - 299.428,892,062,587 \times z \\ -899.142,930,000,000,000,000 + 90.000,000,000,000 \times z \end{array}\right\}$$

$= 98.002,889,762,252,661,333 - 209.428,892,062,587 \times z.$

But $x^3 - 90x$ is $= 98$.

Therefore $98.002,889,762,252,661,333 - 209.428,892,062,587 \times z$ will also be $= 98$. And consequently $98.002,889,762,252,661,333$ will be $= 98 + 209.428,892,062,587 \times z$, and (subtracting 98 from both sides,) $209.428,892,062,587 \times z$ will be $= 0.002,889,762,252,661,333$, and consequently z will be $=$

$$\frac{0.002,889,762,252,661,333}{209.428,892,062,587} = 0.000,013,798.$$

Therefore x, or $9.990,477 - z$, will be $= 9.990,477 - 0.000,013,798 = 9.990,463,202$; that is, the third near value of the root of the proposed equation $x^3 - 90x = 98$ will be $9.990,463,202$.

Q. E. I.

This value of x is probably exact in the first nine figures $9.990,463,20$, or at least in the first eight figures $9.990,463,2$, if no mistake has been made in performing the arithmetical operations by which it has been obtained.

Art. LXII. These six examples will, I presume, be sufficient to explain and illustrate this method of resolving cubick

cubick equations of the general form $x^3 - bx = c$, in the ſecond caſe of it (in which the abſolute term c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$,) by Mr. Raphſon's method of approximation. And this method of reſolving theſe equations is, as I conceive, the eaſieſt and beſt method that can be taken for the purpoſe *. Nothing more therefore need be ſaid concerning the reſolution of theſe equations.

OF THE CUBICK EQUATION $bx - x^3 = c$.

Art. LXIII. BUT there is one more form of cubick equations that involve only the cube and the ſimple power of x, or the unknown quantity, that yet remains to be conſidered. And that is the equation $bx - x^3 = c$.

Now in this equation x^3 is ſubtracted from bx, and conſequently muſt be either leſs than bx, or equal to it, and therefore

* This method of reſolving equations by approximation, which was publiſhed by Mr. Joſeph Raphſon, A. M. in the year 1690, differs but little from that which had been invented for the ſame purpoſe by Sir Iſaac Newton in the year 1666, and which is contained in his very curious little tract intitled *Analyſis per æquationes numero terminorum infinitas*. This tract was communicated by him to Dr. Iſaac Barrow, and by Dr. Barrow, with his leave, to Mr. John Collins and other eminent mathematicians of the Royal Society of London, in the year 1669, but was never printed till the year 1711, or 1712. But an extract

therefore xx muſt be either leſs than, or equal to, the co-efficient b, and x muſt be either leſs than, or equal to, $\sqrt{b}$, or the ſquare-root of the ſaid co-efficient. And, ſince x is limited in it's magnitude ſo as never to be greater than $\sqrt{b}$, it is evident that $bx - x^3$ muſt alſo be limited in it's magnitude, and can never be greater than a certain finite quantity; and therefore, if the abſolute term c of the equation $bx - x^3 = c$ is greater than a certain finite quantity, it is evident that the ſaid equation will be impoſſible, or the propoſition that affirms the compound quantity $bx - x^3$ to be equal to c (for every equation is a propoſition affirming the equality of one quantity, or ſet of quantities, to another,) will be a falſe propoſition. We muſt therefore inquire what is the quantity that is the limit of the magnitude of the compound quantity $bx - x^3$, or than which the ſaid compound quantity can never be greater. Now this inquiry may be made in the following manner:

tract from it, containing a ſpecimen of Sir Iſaac's method of reſolving equations by approximation, had been publiſhed by Dr. Wallis in his Algebra in the year 1685, that is, five years before the publication of Mr. Raphſon's method. The two methods bear a near reſemblance to each other: but of the two I am inclined to give the preference to Mr. Raphſon's method. See a compariſon between theſe two methods in pages 559, 560, &c - - - 590, of a collection of Mathematical Tracts intitled, *The Doctrine of Permutations and Combinations*, which was publiſhed in the year 1795 in a large volume in octavo, and ſold by Mr. White, the bookſeller in Fleet-ſtreet.

OF THE INCREASE AND DECREASE OF THE COMPOUND QUANTITY $bx - x^3$ WHILE x INCREASES FROM o TO $\sqrt{b}$.

Art. LXIV. WHEN x is $= o$, both bx and x^3 will likewiſe be $= o$, and conſequently their difference $bx - x^3$ will alſo be $= o$: and, when x is $= \sqrt{b}$, xx will be $= b$, and x^3 will be $= bx$, and conſequently $bx - x^3$ will be $(= bx - bx) = o$; that is, the compound quantity $bx - x^3$, after having been at firſt equal to o, when x was equal to o, and after having then been equal to ſeveral different finite quantities, while x was of various finite magnitudes between o and $\sqrt{b}$, will, when x is equal to $\sqrt{b}$, become a ſecond time equal to o. There muſt therefore have been ſome finite quantity to which the compound quantity $bx - x^3$ will have increaſed from o, and from which it will have again decreaſed to o, while x has continually increaſed from o to $\sqrt{b}$. This finite quantity, or greateſt poſſible magnitude of the compound quantity $bx - x^3$, we muſt now endeavour to determine.

Let the quantity x be ſuppoſed to increaſe from o to $\sqrt{b}$ by the continual addition of very ſmall equal increments, as, for example, by increments that are equal to only the ten-thouſand millionth part of $\sqrt{b}$; and let each of theſe very ſmall increments of x be denoted by $\dot{x}$, or the letter x with a point placed over it. Then it is evident, that, while x increaſes from x to $x + \dot{x}$, or receives an addition of one of theſe very ſmall increments, the quantity bx will increaſe from bx to $b \times x + \dot{x}$, or $bx + b\dot{x}$, or

will receive the increment $b\dot{x}$; and the quantity x^3 will increaſe from x^3 to $\overline{x + \dot{x}}^3$, or to $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, or will receive the increment $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$. Now this increment is to $b\dot{x}$, or the contemporary increment of bx, as $3x^2 + 3x\dot{x} + \dot{x}\dot{x}$ is to b, or (neglecting the two terms $3x\dot{x}$ and $\dot{x}\dot{x}$ on account of their extream ſmallneſs in compariſon of the firſt term $3x^2$ of the trinomial quantity $3x^2 + 3x\dot{x} + \dot{x}\dot{x}$, to which they are added,) as $3x^2$ is to b. Therefore, while $3xx$ is leſs than b, or xx is leſs than $\frac{b}{3}$, the increment of x^3 will be leſs than the contemporary increment of bx; but, when $3xx$ is greater than b, or xx is greater than $\frac{b}{3}$, (though leſs than b,) the increment of x^3 will be greater than the contemporary increment of bx.

But, while x increaſes from x to $x + \dot{x}$, or receives the increment $\dot{x}$, and bx receives the increment $b\dot{x}$, and x^3 receives the increment $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, or (neglecting the two latter quantities,) $3x^2\dot{x}$, the compound quantity $bx - x^3$ will become equal to $bx + b\dot{x} - x^3 - 3x^2\dot{x} - 3x\dot{x}^2 - \dot{x}^3$, or (neglecting the two terms $3x\dot{x}^2$ and $\dot{x}^3$) $bx + b\dot{x} - x^3 - 3x^2\dot{x}$, or $bx - x^3 + b\dot{x} - 3x^2\dot{x}$, that is, it will be increaſed by $b\dot{x}$, or the increment of bx, and diminiſhed by $3x^2\dot{x}$, or the increment of x^3. Therefore, ſo long as $b\dot{x}$, or the increment of bx, continues to be greater than $3x^2\dot{x}$, or the contemporary increment of x^3, the compound quantity $bx - x^3$ will increaſe; but afterwards, when $b\dot{x}$, or the increment of bx, is leſs than $3x^2\dot{x}$, or the contemporary increment of x^3, it will decreaſe. And therefore the greateſt magnitude of the compound quantity $bx - x^3$ will

will be that which it has when $b\dot{x}$, or the increment of bx, is exactly equal to $3x^2\dot{x}$, or the contemporary increment of x^3, that is, when b is $= 3x^2$, or x^2 is $= \frac{b}{3}$.

But, when xx is $= \frac{b}{3}$, x is $\frac{\sqrt{b}}{\sqrt{3}}$, and x^3 is $= \frac{b\sqrt{b}}{3\sqrt{3}}$, and bx is $= \frac{b\sqrt{b}}{\sqrt{3}}$; and consequently $bx - x^3$ is $= \frac{b\sqrt{b}}{\sqrt{3}} - \frac{b\sqrt{b}}{3\sqrt{3}} \left(= \frac{3b\sqrt{b}}{3\sqrt{3}} - \frac{b\sqrt{b}}{3\sqrt{3}}\right) = \frac{2b\sqrt{b}}{3\sqrt{3}}$.

Therefore the greateſt poſſible magnitude of the compound quantity $bx - x^3$ is $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and conſequently the greateſt poſſible magnitude of c, or the abſolute term of the equation $bx - x^3 = c$, is $\frac{2b\sqrt{b}}{3\sqrt{3}}$.

Q. E. I.

Art. LXV. It follows from the foregoing inveſtigation of the greateſt poſſible magnitude of the compound quantity $bx - x^3$, that, if c, or the abſolute term of the equation $bx - x^3 = c$, is exactly equal to $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or if $\frac{cc}{4}$ is exactly equal to $\frac{b^3}{27}$, the equation $bx - x^3 = c$ will have but one root, and that root will be $\frac{\sqrt{b}}{\sqrt{3}}$; and that, if the abſolute term c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or if $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$, the equation $bx - x^3 = c$ will have two roots, of

which the leſſer will be leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, but leſs than $\sqrt{b}$. Therefore in this equation $bx - x^3 = c$, as well as in the ſecond caſe of the oppoſite equation $x^3 - bx = c$, (in which c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$,) we have very convenient limits to the magnitudes of the roots of the equation, which will enable us to begin a proceſs of approximation to their true values, according to the rules of Mr. Raphſon's method, with great advantage.

Theſe things being premiſed, I ſhall now proceed to ſhew how equations of this form, $bx - x^3 = c$, may be reſolved by Mr. Raphſon's method of approximation, by the Solution of the following Problem:

PROBLEM IV.

Art. LXVI. TO reſolve the cubick equation $bx - x^3 = c$ by Mr. Raphſon's method of approximation, when the abſolute term c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or when $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$.

SOLUTION.

In this equation the abſolute term c is ſuppoſed to be leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, becauſe, if it is exactly equal to $\frac{2b\sqrt{b}}{3\sqrt{3}}$,

or

or if $\frac{cc}{4}$ is exactly equal to $\frac{b^3}{27}$, the equation will have but one root; and that one root, we know by what has been ſhewn in the foregoing articles, will be $= \frac{\sqrt{b}}{\sqrt{3}}$. But, when c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, the equation will have two roots, of which we only know the limits, but not the roots themſelves without an inveſtigation of their values. And of theſe roots it may ſometimes be required to find the greater, and at other times to find the leſſer, according as the greater or the leſſer root may be the quantity that is neceſſary to the Solution of the Problem from which the equation has been derived: and ſometimes it may be neceſſary to find them both. I ſhall therefore ſhew, firſt, how to find the leſſer of theſe two roots by Mr. Raphſon's method of approximation, and, ſecondly, how to find the greater of them by the ſame method.

THE INVESTIGATION OF THE LESSER ROOT OF THE EQUATION $bx - x^3 = c$.

Art. LXVII. SINCE the leſſer root of the equation $bx - x^3 = c$ is leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, and the abſolute term c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, let us chooſe by conjecture ſome quantity leſs than $\frac{\sqrt{b}}{\sqrt{3}}$ for a near value of the ſaid leſſer root, taking

ing it nearer, or leſs near, to $\frac{\sqrt{b}}{\sqrt{3}}$, according as the abſolute term c is nearer, or leſs near, to $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or the greateſt poſſible magnitude of $bx - x^3$; and let us call the quantity ſo choſen a. Then let this quantity a be ſubſtituted inſtead of x in the compound quantity $bx - x^3$, in order to diſcover whether the value of the ſaid compound quantity reſulting from this ſubſtitution will be greater, or will be leſs, than c, or the abſolute term of the equation. And, if the ſaid reſult ſhall be greater than the abſolute term c, we may conclude that a will be greater than the true value of the ſaid leſſer root of the equation; and, if the ſaid reſult ſhall be leſs than the abſolute term c, we may conclude that a will be leſs than the true value of the ſaid leſſer root: becauſe it has been ſhewn above in Art. 64, that the compound quantity $bx - x^3$ increaſes continually while x increaſes from 0 to $\frac{\sqrt{b}}{\sqrt{3}}$. When we have thus diſcovered whether a is greater or is leſs than the leſſer root of the equation, we muſt, if it is greater than the ſaid leſſer root, put $x = a - z$, and ſubſtitute $a - z$ inſtead of x in the equation $bx - x^3 = c$, and reſolve the new equation, reſulting from ſuch ſubſtitution, as if it were a mere ſimple equation, agreeably to Mr. Raphſon's directions; and, if a is leſs than the ſaid leſſer root, we muſt put $x = a + z$, and ſubſtitute $a + z$ inſtead of x in the equation $bx - x^3 = c$, and reſolve the new equation, reſulting from ſuch ſubſtitution, as if it were a mere ſimple equation. And, having thus found the value of z, we muſt, in the firſt caſe, ſubtract it from a, to obtain the value of $a - z$; and, in the ſecond caſe, we

muſt

muſt add it to a, to obtain the value of $a + z$; and the value of $a - z$, or $a + z$, ſo obtained, will be a ſecond near value of x, or the leſſer root of the propoſed equation $bx - x^3 = c$, that will come much nearer than the firſt near value a to it's true value. And by repeating theſe proceſſes of approximation to the true value of x, or the ſaid leſſer root, once or twice more, or oftener, we may obtain the value of the ſaid leſſer root to as great a degree of exactneſs as we pleaſe.

Q. E. I.

THE INVESTIGATION OF THE GREATER ROOT OF THE EQUATION $bx - x^3 = c$.

Art. LXVIII. TO find the value of the greater root of the equation $bx - x^3 = c$ we muſt proceed as follows:

This greater root of the equation $bx - x^3 = c$ is always greater than $\frac{\sqrt{b}}{\sqrt{3}}$, but leſs than $\sqrt{b}$. We may therefore always begin our approximation to it's true value with advantage by chuſing, by conjecture, ſome quantity that is greater than $\frac{\sqrt{b}}{\sqrt{3}}$, but leſs than $\sqrt{b}$, for the firſt near value of it, and calling the quantity ſo choſen a. In making this choice we muſt be governed by a conſideration of the magnitude of c, or the abſolute term of the equation; becauſe we

we know that, when c is equal to $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or when $\frac{cc}{4}$ is $=$ $\frac{b^3}{27}$, this greater root of the equation $bx - x^3 = c$ is equal to $\frac{\sqrt{b}}{\sqrt{3}}$, and that, when c is equal to 0, this greater root of the ſaid equation is equal to $\sqrt{b}$; whence it follows that, if c is but little leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, this greater root of the ſaid equation will be but little greater than $\frac{\sqrt{b}}{\sqrt{3}}$, and that if c is much leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and but little greater than 0, this greater root of the ſaid equation will be but little leſs than $\sqrt{b}$. Therefore, when c is pretty near it's greateſt poſſible magnitude, $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is pretty nearly equal to $\frac{b^3}{27}$, it will be proper to chuſe ſome quantity but little greater than $\frac{\sqrt{b}}{\sqrt{3}}$ for a, or our firſt near value of the greater root of the equation $bx - x^3 = c$; and, when c is much leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and but little greater than 0, it will be proper to chuſe ſome quantity but little leſs than $\sqrt{b}$ for a, or our firſt near value of the greater root of that equation.

Art. LXIX. When we have fixed upon the quantity a for the firſt near value of the greater root of the equation $bx - x^3 = c$, we muſt ſubſtitute it inſtead of x in the compound quantity $bx - x^3$, in order to diſcover whether the value of the ſaid compound quantity reſulting from

from ſuch ſubſtitution will be greater, or will be leſs, than c, or the abſolute term of the equation $bx - x^3 = c$. And, if the ſaid reſult ſhall be greater than c, we muſt conclude that the quantity a will be leſs than the true value of the greater root of the equation $bx - x^3 = c$, becauſe the compound quantity $bx - x^3$ decreaſes continually from $\frac{2b\sqrt{b}}{3\sqrt{3}}$ to 0, while x increaſes from $\frac{\sqrt{b}}{\sqrt{3}}$ to $\sqrt{b}$, as has been ſhewn in Art. 64; and therefore in this caſe we muſt put $x = a + z$, and muſt ſubſtitute $a + z$ inſtead of x in the equation $bx - x^3 = c$, omitting all the terms that involve either the cube or the ſquare of z, agreeably to Mr. Raphſon's directions, and then muſt reſolve the new equation reſulting from ſuch ſubſtitution, (which, with the omiſſions juſt-mentioned, will be a ſimple equation,) by which we ſhall obtain a near value of z; and this near value of z, being added to a, will give us a near value of $a + z$, or x, or a ſecond near value of the greater root of the propoſed equation $bx - x^3 = c$, that will come much nearer than the firſt near value a to the true value of the ſaid root. And on the other hand, if the reſult of the ſubſtitution of a in the compound quantity $bx - x^3$ is leſs than the abſolute term c, we muſt conclude that the quantity a is greater than the true value of x, or the greater root of the equation $bx - x^3 = c$; and therefore in this caſe we muſt put $x = a - z$, and muſt ſubſtitute $a - z$ inſtead of x in the equation $bx - x^3 = c$, omitting all the terms that involve either the cube or the ſquare of z, agreeably to Mr. Raphſon's directions, and then muſt reſolve the new equation reſulting from ſuch ſubſtitution, (which, with the omiſſions juſt-mentioned, will be a ſimple

ple equation,) by which we ſhall obtain a near value of z; and this near value of z, being ſubtracted from a, will give a near value of $a - z$, or x, or a ſecond near value of the greater root of the propoſed equation $bx - x^3 = c$, which will come much nearer than a, the firſt near value of it, to it's true value. And by repeating theſe proceſſes of approximation to the true value of x, or the ſaid greater root of the ſaid equation, once or twice more, or oftener, we may obtain the value of the ſaid greater root to as great a degree of exactneſs as we pleaſe.

Q. E. I.

OF THE RELATION OF THE CUBICK EQUATION $bx - x^3 = c$ TO THE CUBICK EQUATION $x^3 - bx = c$ IN THE SECOND CASE OF THAT EQUATION, WHICH CANNOT BE RESOLVED BY CARDAN'S SECOND RULE.

Art. LXX. IT appears from what has been ſhewn in the foregoing articles, that the limit of the magnitude of the abſolute term c of the cubick equation $bx - x^3 = c$ is the ſame with the limit of the magnitude of the abſolute term c in the ſecond caſe of the oppoſite equation $x^3 - bx = c$, which cannot be reſolved by Cardan's ſecond rule. For in both theſe equations the ſaid limit is $\frac{2b\sqrt{b}}{3\sqrt{3}}$. And therefore, whenever the magnitude of the abſolute term c of

c of the equation $x^3 - bx = c$ is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or is ſuch that the ſaid equation cannot be reſolved by Cardan's ſecond rule, the oppoſite equation $bx - x^3 = c$, which has the ſame quantity e for it's abſolute term, will always be a poſſible equation. This ſeems to be a remarkable point of reſemblance between theſe two equations.

And, further, we may obſerve that, when the abſolute term c of the equation $bx - x^3 = c$ is the greateſt poſſible, or is equal to $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and this equation has but one root, to wit, $\frac{\sqrt{b}}{\sqrt{3}}$, the root of the oppoſite equation $x^3 - bx = c$ will be $\frac{2\sqrt{b}}{\sqrt{3}}$, or exactly double of the ſaid ſingle root of the equation $bx - x^3 = c$: ſo that, when c is equal to this higheſt limit, the knowledge of the root of one of theſe equations will lead us to that of the root of the oppoſite equation.

Art. LXXI. But there is alſo a further connection between theſe two oppoſite equations, by which the knowledge of the roots of the equation $bx - x^3 = c$ may be made the means of our finding the root of the equation $x^3 - bx = c$, when c is leſs than the ſaid limit, or $\frac{2b\sqrt{b}}{3\sqrt{3}}$. For, whatever be the magnitude of the abſolute term c of the equation $bx - x^3 = c$, within the ſaid limit $\frac{2b\sqrt{b}}{3\sqrt{3}}$, the root of the oppoſite equation $x^3 - bx = c$ will always be equal to the ſum of the two roots of the equation

tion $bx - x^3 = c$, as will appear from the demonſtration of the following Theorem :

A THEOREM.

IF c, or the abſolute term of the equation $bx - x^3 = c$, is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, (which is it's greateſt poſſible magnitude,) and the ſaid equation has conſequently two roots, of which the leſſer will be leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, and the greater will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, but leſs than $\sqrt{b}$, the ſum of the ſaid two roots will be equal to the root of the oppoſite equation $x^3 - bx = c$.

DEMONSTRATION.

FOR the ſake of avoiding intricacy and confuſion in the notation of theſe three roots, let us denote the leſſer root of the equation $bx - x^3 = c$ by the letter v, and the greater root of the ſame equation by the letter t, and the root of the oppoſite equation $x^3 - bx = c$ (which has but one root,) by the letter z. And we ſhall then be to demonſtrate that $t + v$ will be equal to z.

Now,

Now, ſince t is a root of the equation $bx - x^3 = c$, we ſhall have $bt - t^3 = c$; and, ſince v is alſo a root of the ſame equation, we ſhall have $bv - v^3 = c$. Therefore $bt - t^3$ will be $= bv - v^3$; and conſequently, (adding t^3 to both ſides,) bt will be $= bv - v^3 + t^3$, or to $bv + t^3 - v^3$.

Further, ſince v is leſs than t, bv will be leſs than bt, and conſequently than $bv + t^3 - v^3$, or the other ſide of the equation; and therefore it may be ſubtracted from both ſides. Let it be ſo ſubtracted. And then we ſhall have $bt - bv = t^3 - v^3$. But $bt - bv$ is $= b \times \overline{t - v}$, and $t^3 - v^3$ is $= \overline{tt + tv + vv} \times \overline{t - v}$. Therefore, (dividing both ſides by $t - v$,) we ſhall have $tt + tv + vv = b$, and conſequently $tt + tv = b - vv$, or $tt + vt = b - vv$. Therefore $tt + vt + \frac{vv}{4}$ will be $= b - vv + \frac{vv}{4}$ $(= b - \frac{3vv}{4}) = \frac{4b - 3vv}{4}$, and (extracting the ſquare-roots of both ſides,) $t + \frac{v}{2}$ will be $= \frac{\sqrt{4b - 3vv}}{2}$; and conſequently t will be $= \frac{\sqrt{4b - 3vv}}{2} - \frac{v}{2}$.

Therefore $t + v$ will be $= \frac{\sqrt{4b - 3vv}}{2} - \frac{v}{2} + v$ $(= \frac{\sqrt{4b - 3vv}}{2} + \frac{v}{2}) = \frac{v + \sqrt{4b - 3vv}}{2}$.

Therefore $\sqrt{t + v}^3$ will be $= \sqrt{\left(\frac{v + \sqrt{4b - 3vv}}{2}\right)^3}$ $(=$

$\sqrt{\frac{v + \sqrt{4b - 3vv}^3}{2^3}} = \sqrt{\frac{v + \sqrt{4b - 3vv}^3}{8}} =$

$\frac{v^3 + 3v^2 \times \sqrt{4b - 3vv} + 3v \times \overline{4b - 3vv} + \overline{4b - 3vv} \times \sqrt{4b - 3vv}}{8}$

$= \frac{v^3 + 12bv - 9v^3 + \overline{4b - 3vv + 3vv} \times \sqrt{4b - 3vv}}{8}$

$= \frac{12bv - 8v^3 + 4b \times \sqrt{4b - 3vv}}{8}) =$

$\frac{3bv - 2v^3 + b \times \sqrt{4b - 3vv}}{2}$; and $b \times \sqrt{t + v}$ will be $=$

$b \times \sqrt{\frac{v + \sqrt{4b - 3vv}}{2}} = \frac{bv + b \times \sqrt{4b - 3vv}}{2}$; and

consequently $\sqrt{t + v}^3 - b \times \sqrt{t + v}$ will be $=$

$\frac{3bv - 2v^3 + b \times \sqrt{4b - 3vv}}{2} - \sqrt{\frac{bv + b \times \sqrt{4b - 3vv}}{2}}$

$= \frac{3bv - 2v^3 + b \times \sqrt{4b - 3vv} - bv - b \times \sqrt{4b - 3vv}}{2}$

$= \frac{2bv - 2v^3}{2} = bv - v^3) = c.$

But $z^3 - bz$ is $= c.$

Therefore $\sqrt{t + v}^3 - b \times \sqrt{t + v}$ is $= z^3 - b \times z$, and consequently $t + v$ muſt be $= z$, or the ſum of the two roots of the equation $bx - x^3 = c$ will be equal to the root of the oppoſite equation $x^3 - bx = c$.

Q. E. D.

Art.

Art. LXXII. It follows from this Propoſition, that, when-ever we have occaſion to reſolve a cubick equation of the form $x^3 - bx = c$, in the ſecond caſe of that equation, or when the abſolute term c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or when $\frac{cc}{4}$ is leſs than $\frac{b^3}{27}$, we might, if there were any convenience attending it, find it's root in an indirect manner, by, firſt reſolving the oppoſite equation $bx - x^3 = c$, and finding both it's roots, and then adding the ſaid roots together; ſince their ſum would be equal to the root of the other equation $x^3 - bx = c$, which we are ſeeking. But this way of obtaining that root would, for the moſt part, be more troubleſome than to compute it directly, by Mr. Raphſon's method of approximation, in the manner above-explained: though, perhaps, there may now and then be caſes in which ſuch an indirect method of reſolving an equation of that form might be uſeful.

OF THE REDUCTION OF THE CUBICK EQUATION $bx - x^3 = c$ TO A QUADRATICK EQUATION, WHEN EITHER OF IT'S TWO ROOTS IS KNOWN.

Art. LXXIII. WE may alſo obſerve, that, when one of the two roots of the equation $bx - x^3 = c$ has been computed in the manner above explained, or is known by

any other means, the equation may be reduced to a quadratick equation of which the root will be equal to the other root of the cubick equation from which it is derived. This may be ſhewn in the following manner.

Let us, firſt, ſuppoſe the leſſer root v of the cubick equation $bx - x^3 = c$ to be known, and the greater root t to be ſought.

Then, ſince both v and t are roots of the equation $bx - x^3 = c$, and may conſequently be ſubſtituted in that equation inſtead of x, we ſhall have $bv - v^3 = c$, and $bt - t^3 = c$, and conſequently $bv - v^3 = bt - t^3$. Therefore bt will be $= bv - v^3 + t^3$, or $bv + t^3 - v^3$, and $bt - bv$ will be $= t^3 - v^3$, and $\frac{bt - bv}{t - v}$ will be $= \frac{t^3 - v^3}{t - v}$. But $\frac{bt - bv}{t - v}$ is $= b$, and $\frac{t^3 - v^3}{t - v}$ is $= tt + tv + vv$. Therefore b will be $= tt + tv + vv$, which is a quadratick equation involving the greater, or unknown, root t of the cubick equation $bx - x^3 = c$ together with the known quantities b and v. Therefore, if we reſolve this quadratick equation $tt + tv + vv = b$, or $tt + vt + vv = b$, we ſhall thereby obtain the value of t, or the greater, or unknown, root of the original equation $bx - x^3 = c$. This reſolution may be performed as follows. Since $tt + vt + vv$ is $= b$, if we ſubtract vv from both ſides, we ſhall have $tt + vt = b - vv$.

Therefore (adding $\frac{vv}{4}$ to both ſides,) we ſhall have

$tt +$

$tt + vt + \frac{vv}{4} = b - vv + \frac{vv}{4} \left(= b - \frac{3vv}{4}\right) = \frac{4b - 3vv}{4}$.

Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $t + \frac{v}{2} = \frac{\sqrt{4b - 3vv}}{2}$, and conſequently

$t = \frac{\sqrt{4b - 3vv}}{2} - \frac{v}{2} = \frac{\sqrt{4b - 3vv} - v}{2}$.

Q. E. I.

And, ſecondly, let us ſuppoſe the greater root t of the equation $bx - x^3 = c$ to be known, and the leſſer root v to be ſought.

Then, ſince $bv - v^3$ is $= bt - t^3$, and conſequently $bv + t^3 - v^3$ is $= bt$, and $t^3 - v^3$ is $= bt - bv$, we ſhall have $\frac{t^3 - v^3}{t - v} = \frac{bt - bv}{t - v}$, that is, $tt + tv + vv = b$; which is a quadratick equation involving the leſſer, or unknown, root v of the cubick equation $bx - x^3 = c$ together with the known quantities b and t. Therefore, if we reſolve this quadratick equation $tt + tv + vv = b$, or $vv + tv + tt = b$, we ſhall thereby obtain the value of v, or the leſſer, or unknown, root of the original equation $bx - x^3 = c$. This reſolution may be performed as follows. Since $vv + tv + tt$ is $= b$, by ſubtracting tt from both ſides we ſhall have $vv + tv = b - tt$, and (by adding $\frac{tt}{4}$ to both ſides,) $vv + tv + \frac{tt}{4} \left(= b - tt + \frac{tt}{4} = b - \frac{3tt}{4}\right) = \frac{4b - 3tt}{4}$. Therefore by extracting the

ſquare-roots of both ſides,) we ſhall have $v + \frac{t}{2} = \frac{\sqrt{4b - 3tt}}{2}$, and conſequently $v = \frac{\sqrt{4b - 3tt}}{2} - \frac{t}{2}$, or $\frac{\sqrt{4b - 3tt} - t}{2}$.

Q. E. I.

And this way of finding the ſecond root of the cubick equation $bx - x^3 = c$, when one of it's roots is already known, which requires only the computation of the expreſſion $\frac{\sqrt{4b - 3vv} - v}{2}$ or the expreſſion $\frac{\sqrt{4b - 3tt} - t}{2}$, will be eaſier than the method of finding it given above in the Solution of the foregoing, or 4th, Problem, which requires the application of Mr. Raphſon's method of approximation, and ought therefore to be preferred to it. And conſequently there will never be occaſion to make uſe of that method for more than one of the roots of the cubick equation $bx - x^3 = c$.

OF

OF THE REDUCTION OF THE CUBICK EQUATION $x^3 - bx = c$, IN THE SECOND CASE OF IT, OR WHEN c IS LESS THAN $\frac{2b\sqrt{b}}{3\sqrt{3}}$, OR $\frac{cc}{4}$ IS LESS THAN $\frac{b^3}{27}$, TO A QUADRATICK EQUATION, WHEN EITHER OF THE TWO ROOTS OF THE OPPOSITE EQUATION $bx - x^3 = c$ IS KNOWN.

Art. LXXIV. WE have ſeen in the laſt article, that, if either of the two roots of the cubick equation $bx - x^3 = c$ is known, the other may be derived from it by reſolving a quadratick equation. But we muſt likewiſe obſerve that, when either of thoſe two roots is known, the root of the oppoſite equation $x^3 - bx = c$ may alſo be derived from it by reſolving a quadratick equation, without computing the other root of the equation $bx - x^3 = c$. This may be ſhewn in the manner following.

Let us ſuppoſe that v, or the leſſer of the two roots of the equation $bx - x^3 = c$ is known, and that we are required to find z, or the root of the oppoſite equation $x^3 - bx = c$, or $z^3 - bz = c$.

Then, ſince $bv - v^3$ is $= c$, and $z^3 - bz$ is alſo $= c$, we ſhall have $z^3 - bz = bv - v^3$, and conſequently (adding bz to both ſides,) $z^3 = bv - v^3 + bz$, and (adding

v^3 to both ſides,) $z^3 + v^3 = bv + bz$, or $z^3 + v^3 = bz + bv$. Therefore $\frac{z^3 + v^3}{z + v}$ will be $= \frac{bz + bv}{z + v}$. But $\frac{z^3 + v^3}{z + v}$ is $= zz - zv + vv$, and $\frac{bz + bv}{z + v}$ is $= b$. Therefore $zz - zv + vv$ will be $= b$; which is a quadratick equation involving the unknown root z of the equation $z^3 - bz = c$ together with the known quantities b and v. Therefore by reſolving this quadratick equation $zz - zv + vv = b$ we may obtain the value of z, or the root of the cubick equation $z^3 - bz = c$, without firſt finding the value of t, or the greater root of the equation $bx - x^3 = c$. This reſolution may be performed as follows. Subtract vv from both ſides of the equation $zz - zv + vv = b$; and we ſhall have $zz - zv = b - vv$, or $zz - vz = b - vv$. Therefore, (adding $\frac{vv}{4}$ to both ſides,) we ſhall have $zz - zv + \frac{vv}{4}$ $(= b - vv + \frac{vv}{4} = b - \frac{3vv}{4})$ $= \frac{4b - 3vv}{4}$. Therefore, (extracting the ſquare-roots of both ſides,) we ſhall have $z - \frac{v}{2} = \frac{\sqrt{4b - 3vv}}{2}$, and conſequently $z = \frac{\sqrt{4b - 3vv}}{2} + \frac{v}{2}$, or $\frac{v + \sqrt{4b - 3vv}}{2}$.

Q. E. I.

Let us next ſuppoſe that t, or the greater of the two roots of the equation $bx - x^3 = c$, is known, and that we

are

are required to find z, or the root of the oppoſite $x^3 - bx = c$, or $z^3 - bx = c$, as before.

Then, ſince $bt - t^3$ is $= c$, and $z^3 - bz$ is alſo $= c$, we ſhall have $z^3 - bz = bt - t^3$, and (adding bz to both ſides,) $z^3 = bt - t^3 + bz$, and (adding t^3 to both ſides) $z^3 + t^3 = bz = bt$. Therefore $\frac{z^3 + t^3}{z + t}$ will be $= \frac{bz + bt}{z + t}$. But $\frac{z^3 + t^3}{z + t}$ is $= zz - zt + tt$, and $\frac{bz + bt}{z + t}$ is $= b$. Therefore $zz - zt + tt$ will be $= b$; which is a quadratick equation involving the unknown root z of the equation $z^3 - bz = c$ together with the known quantities b and t. Therefore by reſolving this equation $zz - zt + tt = b$ we may obtain the value of z, or the root of the cubick equation $z^3 - bz = c$ without firſt finding the value of v, or the leſſer root of the equation $bx - x^3 = c$. This reſolution may be performed as follows. Subtract tt from both ſides of the equation $zz - zt + tt = b$; and we ſhall have $zz - zt = b - tt$, or $zz - tz = b - tt$. Therefore (adding $\frac{tt}{4}$ to both ſides,) we ſhall have $zz - tz + \frac{tt}{4}$ $(= b - tt + \frac{tt}{4} = b - \frac{3tt}{4}) = \frac{4b - 3tt}{4}$, and (extracting the ſquare-roots of both ſides,) $z - \frac{t}{2} = \frac{\sqrt{4b - 3tt}}{2}$, and conſequently $z = \frac{\sqrt{4b - 3tt}}{2} + \frac{t}{2}$, or $\frac{t + \sqrt{4b - 3tt}}{2}$.

Q. E. I.

Therefore,

Therefore, when the two roots t and v of the equation $bx - x^3 = c$ are known, we may obtain the root z of the oppoſite equation $x^3 - bx = c$, or $z^3 - bz = c$, by computing either of the expreſſions $\frac{t + \sqrt{4b - 3tt}}{2}$ and $\frac{v + \sqrt{4b - 3vv}}{2}$.

Art. LXXV. The firſt of theſe expreſſions of the value of z, or the root of the equation $x^3 - bx = c$, to wit, $\frac{t + \sqrt{4b - 3tt}}{2}$, is the ſum of the two roots t and v of of the oppoſite equation $bx - x^3 = c$ expreſſed in terms involving only t; and the ſecond expreſſion of the value of z, to wit, $\frac{v + \sqrt{4b - 3vv}}{2}$, is the ſum of the ſame two roots expreſſed in terms involving only v. For v is $= \frac{\sqrt{4b - 3tt} - t}{2}$, and conſequently $v + t$ is $= \frac{\sqrt{4b - 3tt} - t}{2} + t \left(= \frac{\sqrt{4b - 3tt} - t}{2} + \frac{2t}{2}\right) = \frac{\sqrt{4b - 3tt} + t}{2}$, or $\frac{t + \sqrt{4b - 3tt}}{2}$; and t is $= \frac{\sqrt{4b - 3vv} - v}{2}$, and conſequently $t + v$ is $= \frac{\sqrt{4b - 3vv} - v}{2} + v \left(= \frac{\sqrt{4b - 3vv} - v}{2} + \frac{2v}{2}\right) = \frac{\sqrt{4b - 3vv} + v}{2}$ or $\frac{v + \sqrt{4b - 3vv}}{2}$.

Q. E. D.

OF

OF THE REDUCTION OF THE CUBICK EQUATION $bx - x^3 = c$ TO A QUADRATICK EQUATION, WHEN THE ROOT OF THE OPPOSITE EQUATION $x^3 - bx = c$ IS KNOWN.

Art. LXXVI. IF we are required to find the two roots t and v of the cubick equation $bx - x^3 = c$, and we already know the root z of the oppoſite equation $x^3 - bx = c$, we may reduce the given equation $bx - x^3 = c$ to a quadratick equation of the third form, to wit, $ax - xx = b$, which has two roots; and the roots of the quadratick equation obtained by ſuch reduction will be the roots of the given cubick equation $bx - x^3 = c$.

For, if the leſſer root of the given equation $bx - x^3 = c$ be, for the ſake of diſtinctneſs, called v, and the greater root of it be called t, and the root of the oppoſite equation $x^3 - bx = c$ be called z, (as in the foregoing articles,) we ſhall have $z^3 - bz\ (= x^3 - bx) = c$, and $bt - t^3\ (= bx - x^3) = c$, and conſequently $bt - t^3 = z^3 - bz$. Therefore $bt - t^3 + bz$ will be $= z^3$, and $bt + bz$ will be $= z^3 + t^3$, or $bz + bt$ will be $= z^3 + t^3$. Therefore $\frac{bz + bt}{z + t}$ will be $= \frac{z^3 + t^3}{z + t}$. But $\frac{bz + bt}{z + t}$ is $= b$, and $\frac{z^3 + t^3}{z + t}$ is $= zz - zt + tt$, as will appear by making theſe diviſions. Therefore b will be $= zz - zt + tt$; and conſequently $b + zt$ will be $zz + tt$, and $b + zt - tt$ will be $= zz$, and $zt - tt$ will be $= zz - b$. But $zt - tt$ is

is equal to $t \times \overline{z-t}$, or to the product of the two unequal parts t and $z - t$ of the quantity z, and consequently (by Euclid's Elements, Book II, Prop. 5,) will be less than $\frac{zz}{4}$, or the square of $\frac{z}{2}$, or of half the said quantity z. Therefore $zt - tt$ may be subtracted from $\frac{zz}{4}$. Let us now subtract $zt - tt$, and, it's equal, $zz - b$ from $\frac{zz}{4}$; and we shall have $\frac{zz}{4} - \overline{zt - tt} = \frac{zz}{4} - \overline{zz - b}$. But $\frac{zz}{4} - \overline{zt - tt}$ is $= \frac{zz}{4} - zt + tt$, and $\frac{zz}{4} - \overline{zz - b}$ is $= \frac{zz}{4} - zz + b$. Therefore $zz - zt + tt$ will be equal to $\frac{zz}{4} - zz + b$, or to $\frac{zz - 4zz + 4b}{4}$, or to $\frac{4b - 3zz}{4}$; and consequently the square root of the trinomial quantity $\frac{zz}{4} - zt + tt$ will be equal to the square-root of $\frac{4b - 3zz}{4}$, or to $\frac{\sqrt{4b - 3zz}}{2}$. But the square-root of the trinomial quantity $\frac{zz}{4} - zt + tt$ is $\frac{z}{2} - t$, if $\frac{z}{2}$ is greater than t; and it is $t - \frac{z}{2}$, if t is greater than $\frac{z}{2}$. Now, because t is the greater of the two roots, t and v, of the equation $bx - x^3 = c$, it must be greater than $\frac{t + v}{2}$. But it has been shewn in Art. 71, that

that z is $= t + v$. Therefore $\frac{z}{2}$ muſt be $= \frac{t+v}{2}$; and conſequently t, which is greater than $\frac{t+v}{2}$, muſt alſo be greater than $\frac{z}{2}$. Therefore the ſquare-root of the trinomial quantity $\frac{zz}{4} - zt + tt$ will be $t - \frac{z}{2}$; and conſequently $t - \frac{z}{2}$ will be $= \frac{\sqrt{4b - 3zz}}{2}$. Therefore t, or the greater root of the equation $bx - x^3 = c$, will be $= \frac{\sqrt{4b - 3zz}}{2} + \frac{z}{2}$, or $\frac{\sqrt{4b - 3zz} + z}{2}$, or $\frac{z + \sqrt{4b - 3zz}}{2}$.

Q. E. I.

In like manner we may derive from z, or the root of the equation $z^3 - bz = c$, or $x^3 - bx = c$, an expreſſion for v, or the leſſer of the two roots of the equation $bx - x^3 = c$, by the reſolution of a quadratick equation. This may be done by proceeding as follows:

Since $z^3 - bz$ is $= c$, and $bv - v^3$ is alſo $= c$, we ſhall have $z^3 - bz = bv - v^3$; and conſequently (adding v^3 to both ſides,) $z^3 - bz + v^3$ will be $= bv$, and (adding bz to both ſides,) $z^3 + v^3$ will be $= bv + bz$, or $bz + bv$. Therefore $\frac{z^3 + v^3}{z + v}$ will be $= \frac{bz + bv}{z + v}$. But $\frac{z^3 + v^3}{z + v}$ is $= zz - zv + vv$, and $\frac{bz + bv}{z + v}$ is $= b$. Therefore $zz - zv + vv$ will be $= b$. Therefore $zz +$

vv

vv will be $= b + zv$, and zz will be $= b + zv - vv$, and $zz - b$ will be $= zv - vv$. But $zv - vv$ is equal to $v \times \overline{z - v}$, and conſequently (by El. II, Prop. 5,) is leſs than $\frac{z}{2} \times \frac{z}{2}$, or $\frac{zz}{4}$, and conſequently may be ſubtracted from it. Let $zv - vv$, and, it's equal, $zz - b$, be, both of them, ſubtracted from $\frac{zz}{4}$; and then we ſhall have $\frac{zz}{4} - \overline{zv - vv} = \frac{zz}{4} - \overline{zz - b}$, that is, $\frac{zz}{4} - zv + vv$ will be $= \frac{zz}{4} - zz + b = \frac{zz - 4zz + 4b}{4} = \frac{4b - 3zz}{4}$. Therefore the ſquare-root of the trinomial quantity $\frac{zz}{4} - zv + vv$ will be $= \frac{\sqrt{4b - 3zz}}{2}$. But, becauſe v is leſs than $\frac{z}{2}$, the ſquare-root of the trinomial quantity $\frac{zz}{4} - zv + vv$ will be $\frac{z}{2} - v$. Therefore we ſhall have $\frac{z}{2} - v = \frac{\sqrt{4b - 3zz}}{2}$, and conſequently $\frac{z}{2} = \frac{\sqrt{4b - 3zz}}{2} + v$, and $v = \frac{z}{2} - \frac{\sqrt{4b - 3zz}}{2}$, or $\frac{z - \sqrt{4b - 3zz}}{2}$; that is, v, or the leſſer root of the equation $bx - x^3 = c$, will be $= \frac{z - \sqrt{4b - 3zz}}{2}$.

Q. E. I.

Art. LXXVII. It appears therefore that t, or the greater root of the equation $bx - x^3 = c$, will be $= z +$

$\frac{z + \sqrt{4b - 3zz}}{2}$, and that v, or the leſſer root of the ſame equation, will be $= \frac{z - \sqrt{4b - 3zz}}{2}$. And conſequently, whenever it happens that z, or the root of the equation $z^3 - bz = c$, or the equation $x^3 - bx = c$, is known, we need not, in order to obtain the roots of the equation $bx - x^3 = c$, have recourſe to the method deſcribed above in Problem iv, but may find the ſaid roots by computing the two expreſſions $\frac{z + \sqrt{4b - 3zz}}{2}$ and $\frac{z - \sqrt{4b - 3zz}}{2}$, which will be much leſs troubleſome than inveſtigating their values by the method deſcribed in Problem iv. But, when it is required to find the roots of the equation $bx - x^3 = c$ without knowing the value of z, or the root of the oppoſite equation $z^3 - bz = c$, or $x^3 - bx = c$, it will then be moſt convenient to inveſtigate the values of one of it's two roots by the method deſcribed in Problem iv, and afterwards to find the value of the other root by the expreſſion given for it in Art. 73, which was derived from a quadratick equation. Of the manner of inveſtigating one of the roots of the equation $bx - x^3 = c$ deſcribed in Problem iv, I ſhall now proceed to give a few examples.

EXAM-

EXAMPLES OF THE RESOLUTION OF THE CUBICK EQUATION $bx - x^3 = c$ BY THE METHOD DESCRIBED IN PROBLEM IV.

EXAMPLE I.

Art. LXXVIII. LET it be required to resolve the cubick equation $50x - x^3 = 120$ by the method described in Problem iv.

Here b is $= 50$; c is $= 120$, $\frac{c}{2}$ is $= 60$; $\frac{cc}{4}$ is $= 3600$; b^3 is $= 125{,}000$; and $\frac{b^3}{27}$ is $= \frac{125{,}000}{27} = 4629.629{,}629{,}629{,}629$, &c, which is greater than 3600, or $\frac{cc}{4}$. Therefore cc is less than $\frac{4b^3}{27}$, and c is less than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and consequently the equation $50x - x^3 = 120$ will be a possible equation, and will have two roots, of which the lesser will be less than $\frac{\sqrt{b}}{\sqrt{3}}$ (or $\sqrt{\frac{b}{3}}$, or $\sqrt{\frac{50}{3}}$, or $\sqrt{16.666{,}666}$, &c,) or 4.082, &c, and the greater will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, or 4.082, &c, but less than $\sqrt{b}$, or $\sqrt{50}$,

$\sqrt{50}$, or 7.071, &c. This greater root may be inveſtigated in the following manner:

Art. LXXIX. When x is $= \frac{\sqrt{b}}{\sqrt{3}}$, or 4.082, &c, the compound quantity $bx - x^3$ will be $= \frac{2b\sqrt{b}}{3\sqrt{3}} \left(= 2 \times \frac{b}{3} \times \frac{\sqrt{b}}{\sqrt{3}} = 2 \times \frac{b}{3} \times 4.082 \text{ &c} = 2 \times \frac{50}{3} \times 4.082 \text{ &c} = \frac{100}{3} \times 4.082 \text{ &c} = \frac{408.2 \text{ &c}}{3}\right) = 136.0$ &c; and when x is $= \sqrt{b}$, or $\sqrt{50}$, or 7.071, &c, the ſame compound quantity $bx - x^3$ will be $= 0$. But in the preſent equation $bx - x^3$ is $= 120$, which is much nearer to 136.0 &c than to 0. Therefore it ſeems reaſonable to ſuppoſe that x will be nearer to it's lower limit 4.082 &c than to it's higher limit 7.071, &c, and conſequently that it will be leſs than half the ſum of the ſaid two limits, or than $\frac{4.082 \text{ &c} + 7.071 \text{ &c}}{2}$, or than $\frac{11.153 \text{ &c}}{2}$, or than 5.576, &c. We may therefore conjecture that it is nearly equal to 5, and will therefore ſubſtitute 5 inſtead of it in the compound quantity $50x - x^3$, in order to diſcover whether the reſult of the ſaid ſubſtitution will be greater, or leſs, than 120, or the abſolute term of the equation $50x - x^3 = 120$, and conſequently whether the number 5 will be leſs, or greater, than the true value of the greater root of that equation.

Now, if x be $= 5$, we ſhall have $x^3 = 125$, and $50x \left(= 50 \times 5\right) = 250$, and conſequently $50x - x^3 \left(= 250 - 125\right) = 125$; which is ſomewhat greater than 120, or the abſolute term of the equation $50x - x^3 = 120$.

Therefore 5 muſt be ſomewhat leſs than the true value of the greater root of the ſaid equation. It ſeems, however, to be near enough to the ſaid true value to be made the baſis of a further approximation to the ſaid true value according to the rules of Mr. Raphſon's method.

Art. LXXX. We will therefore ſuppoſe x to be $= 5 + z$, and ſubſtitute this binomial quantity inſtead of x in the propoſed equation $50x - x^3 = 120$, omitting all the terms that involve either the cube or the ſquare of z. This may be done as follows:

Since x is $= 5 + z$, we ſhall have

$$x^3 = \overline{5+z}|^3 \left(= \overline{5}|^3 + 3 \times \overline{5}|^2 \times z + \&c = 125 + 3 \times 25 \times z + \&c\right)$$
$$= 125 + 75 \times z + \&c,$$

and $50x = 50 \times \overline{5+z} \left(= 50 \times 5 + 50 \times z\right) = 250 + 50z$,

and conſequently $50x - x^3 = \left\{\begin{matrix} 250 + 50z \\ -125 - 75z \end{matrix}\right\} =$

$$125 - 25z.$$

But $50x - x^3$ is $= 120$.

Therefore $125 - 25z$ will alſo be $= 120$.

Therefore 125 will be $= 120 + 25z$, and $25z$ will be $= 125 - 120 = 5$, and z will be $= \frac{5}{25} = 0.2$.

Therefore x, or $5 + z$, will be $= 5 + 0.2$, or 5.2; or

5 5.2

5.2 will be a second near value of the greater root of the propoſed equation $50x - x^3 = 120$.

Q. E. I.

Art. LXXXI. Now let 5.2 be ſubſtituted inſtead of x in the compound quantity $50x - x^3$. And we ſhall have $x^3 (= \overline{5.2}^3) = 140.608$, and $50x = 50 \times 5.2 = 260.0$, and conſequently $50x - x^3 (= 260.000 - 140.608) = 119.392$; which is a little leſs than 120, or the abſolute term of the equation $50x - x^3 = 120$. Therefore 5.2 muſt be a little greater than the true value of the greater root of that equation.

In order therefore to obtain a more exact value of this root, let us ſuppoſe it to be $= 5.2 - z$.

Then we ſhall have $x^3 = \overline{5.2 - z}^3 (= 5.2^3 - 3 \times 5.2^2 \times z + \&c = 140.608 - 3 \times 27.04 \times z + \&c) = 140.608 - 81.12 \times z + \&c$,

and $50x = 50 \times \overline{5.2 - z} (= 50 \times 5.2 - 50 \times z) = 260.0 - 50z$,

and conſequently $50x - x^3 = 260.000 - 50z$

$$- \overline{140.608 - 81.12 \times z} = \left\{ \begin{array}{l} 260.000 - 50z \\ - 140.608 + 81.12 \times z \end{array} \right\}$$

$= 119.392 + 31.12 \times z$.

But $50x - x^3$ is $= 120$.

Therefore $119.392 + 31.12 \times z$ will alſo be $= 120$. And conſequently $31.12 \times z$ will be $(= 120 - 119.392)$

$=$

$= 0.608$, and z will be $= \frac{0.608}{31.12} = 0.019$. Therefore x, or $5.2 - z$, will be $= 5.2 - 0.019 = 5.181$; or 5.181 will be a third near value of the greater root of the propoſed equation $50x - x^3 = 120$.

Q. E. I.

This value of x will be pretty near the truth. For, if we ſuppoſe x to be equal to 5.181, we ſhall have $x^3 (= \overline{5.181}^3) = 139.072,344,741$, and $50x (= 50 \times 5.181) = 259.050$, and conſequently

$$50x - x^3 = \left\{\begin{array}{r} 259.050,000,000 \\ - \ 139.072,344,741 \end{array}\right\}$$

$= 119.977,655,259$, which is leſs than 120, or the abſolute term of the equation $50x - x^3 = 120$, by only the ſmall quantity $0.022,344,741$, which is leſs than the 5370th part of 120. Therefore 5.181 will be very nearly equal to, but ſomewhat greater than, the true value of the greater root of the equation $50x - x^3 = 120$.

Art. LXXXII. To find this root to a ſtill greater degree of exactneſs we will therefore ſuppoſe x to be $= 5.181 - z$, and will ſubſtitute this binomial quantity inſtead of x in the equation $50x - x^3 = 120$, omitting all the terms that involve either the cube, or the ſquare, of z.

Then, ſince x is $= 5.181 - z$, we ſhall have

$$x^3 = \overline{5.181 - z}^3 (= \overline{5.181}^3 - 3 \times \overline{5.181}^2 \times z + \&c$$
$$\overline{5.181}^3 - 3 \times 26.842,761 \times z + \&c)$$

=

$$= 139.072,344,741 - 80.528,283 \times z + \&c,$$

and $50x = 50 \times \overline{5.181 - z}\ (= 50 \times 5.181 - 50 \times z$

$$= 259.050 - 50z,$$

and consequently $50x - x^3 = 259.050,000,000 - 50z$

$$- \overline{139.072,344,741 - 80.528,283 \times z} =$$

$$\left\{\begin{matrix} 259.050,000,000 - 50z \\ -139.072,344,741 + 80.528,283 \times z \end{matrix}\right\} =$$

$$119.977,655,259 + 30.528,283 \times z.$$

But $50x - x^3$ is $= 120$.

Therefore $119.977,655,259 + 30.528,283 \times z$ will alſo be $= 120$; and conſequently $30.528,283 \times z$ will be $= 120 - 119.977,655,259 = 0.022,344,741$, and z will be $= \frac{0.022,344,741}{30.528,283} = 0.000,731,9$. Therefore x, or $5.181 - z$, will be $= 5.181 - 0.000,731,9 = 5.180,268,1$; that is, $5.180,268,1$ will be a fourth near value of the greater root of the propoſed equation $50x - x^3 = 120$.

Q. E. I.

This value, $5.180,268,1$, of the greater root of the equation $50x - x^3 = 120$ is probably exact in the firſt ſeven figures $5.180,268$.

Art. LXXXIII. Having thus found $5.180,268$ for the value of t, or the greater root of the propoſed equation $50x - x^3 = 120$, we will derive the leſſer root y from it by computing the expreſſion $\frac{\sqrt{4b - 3tt} - t}{2}$, to which it has been ſhewn above, in Art. 73, to be equal.

Now, since t is $= 5.180,268$, we shall have $tt = 26.835,176,551,824$, and consequently $3tt$ ($= 3 \times 26.835,176,551,824$) $= 80.505,529,655,472$, and $4b - 3tt$ ($= 4 \times 50 - 3tt$) $= 200.000,000,000,000 - 80.505,529,655,472 = 119.494,470,344,528$, and $\sqrt{4b - 3tt} = \sqrt{119.494,470,344,528} = 10.931,352$, and $\sqrt{4b - 3tt} - t = 10.931,352 - 5.180,268 = 5.751,084$, and $\frac{\sqrt{4b - 3tt} - t}{2} = \frac{5.751,084}{2} = 2.875,542$. Therefore v, or the lesser root of the proposed equation $50x - x^3 = 120$, will be $= 2.875,542$.

Q. E. I.

Art. LXXXIV. If these two values of t and v, the roots of the proposed equation $50x - x^3 = 120$, have been found with exactness, their sum ought to give us the value of the root of the opposite equation $x^3 - 50x = 120$. And this we shall find, upon trial, to be the case. For $5.180,268 + 2.875,542$ is $= 8.055,810$, which was found above, in Art. 34, to be the root of the equation $x^3 - 50x = 120$. And this agreement is a confirmation of the justness of the resolutions that have been given of both these equations.

I now proceed to give other examples of the resolution of equations of the form $bx - x^3 = c$ by the method described in Problem iv.

EXAMPLE

EXAMPLE II.

Art. LXXXV. LET the equation to be reſolved be $5x - x^3 = 4$.

Here b is $= 5$, and c is $= 4$; and conſequently $\frac{c}{2}$ is $= 2$, and $\frac{cc}{4}$ is $= 4$, and $\frac{b^3}{27}$ is $= \frac{125}{27} = 4.629,629,629,629$, &c, which is greater than 4, or $\frac{cc}{4}$. Therefore cc is leſs than $\frac{4b^3}{27}$, and conſequently c is leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, and therefore the equation $5x - x^3 = 4$ will be a poſſible equation, and will have two roots, of which the leſſer will be leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, (or $\sqrt{\frac{b}{3}}$, or $\sqrt{\frac{5}{3}}$, or $\sqrt{1.666,666,666,666}$, &c,) or 1.290, &c, and the greater will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, or 1.290, &c, but leſs than $\sqrt{b}$, or $\sqrt{5}$, or 2.236, &c. This greater root may be inveſtigated in the following manner:

Art. LXXXVI. When x is $= \frac{\sqrt{b}}{\sqrt{3}}$, or 1.290, &c, the compound quantity $bx - x^3$ will be $= \frac{2b\sqrt{b}}{3\sqrt{3}}$, (or $\frac{2b}{3} \times \frac{\sqrt{b}}{\sqrt{3}}$, or $\frac{2b}{3} \times 1.290$ &c, or $\frac{2 \times 5}{3} \times 1.290$, &c, or $\frac{10}{3} \times 1.290$, &c, or $\frac{12.90 \text{ &c}}{3}$,) or 4.30 &c; and,

when x is $= \sqrt{b}$, or $\sqrt{5}$, or 2.236, &c, the ſame compound quantity will be $= 0$. But in the preſent caſe the ſaid compound quantity is equal to 4, which is nearer to 4.30 &c, or it's magnitude when x is $= \frac{\sqrt{b}}{\sqrt{3}}$ or, 1,290, &c, than to 0, or it's magnitude when x is $= \sqrt{b}$, or 2.236, &c. Therefore it ſeems reaſonable to ſuppoſe that the greater value of x in the propoſed equation $5x - x^3 = 4$ will be nearer to it's lower limit, $\frac{\sqrt{b}}{\sqrt{3}}$, or 1.290, &c, than to it's higher limit 2.236, &c, and conſequently that it will be leſs than half the ſum of the ſaid two limits, (or than $\frac{1.290 \text{ &c} + 2.236 \text{ &c}}{2}$, or than $\frac{3.526}{2}$,) or than 1.763 &c. We will therefore conjecture that it is nearly equal to 1.5, and will ſubſtitute the ſaid number 1.5 inſtead of it in the compound quantity $5x - x^3$, in order to diſcover whether the reſult of the ſaid ſubſtitution will be greater, or leſs, than 4, or the abſolute term of the equation $5x - x^3 = 4$, and conſequently whether the ſaid number 1.5 is leſs, or greater, than the true value of x, or the greater root of the ſaid equation.

Now, if x is $= 1.5$, we ſhall have $x^3 (= \overline{1.5}^3) = 3.375$, and $5x (= 5 \times 1.5) = 7.5$, and conſequently $5x - x^3 (= 7.500 - 3.375) = 4.125$, which is ſomewhat greater than 4, or the abſolute term of the equation $5x - x^3 = 4$. Therefore 1.5 will be ſomewhat leſs than the true value of the greater root of the ſaid equation. It will, however, be near enough to the ſaid true value to be made the baſis of a further approximation to the ſaid true value according to the rules of Mr. Raphſon's method.

Art.

Art. LXXXVII. We will therefore ſuppoſe x to be $= 1.5 + z$, and will ſubſtitute this binomial quantity inſtead of x in the propoſed equation $5x - x^3 = 4$, omitting all the terms that involve either the cube, or the ſquare, of z, and then will reſolve the new equation reſulting from ſuch ſubſtitution.

Now, ſince x is $= 1.5 + z$, we ſhall have

$x^3 = \overline{1.5 + z}^3$ ($= \overline{1.5}^3 + 3 \times \overline{1.5}^2 \times z$ + &c

$= 3.375 + 3 \times 2.25 \times z$ + &c) $= 3.375 + 6.75 \times z$ + &c,

and $5x = 5 \times \overline{1.5 + z}$ ($= 5 \times 1.5 + 5 \times z$) $= 7.5 + 5z$,

and conſequently $5x - x^3$ ($= 7.5 + 5z - 3.375 - 6.75 \times z$) $= 4.125 - 1.75 \times z$.

But $5x - x^3$ is $= 4$.

Therefore $4.125 - 1.75 \times z$ will alſo be $= 4$. And conſequently 4.125 will be $= 4 + 1.75 \times z$, and $1.75 \times z$ will be ($= 4.125 - 4$) $= 0.125$. Therefore z will be $= \frac{0.125}{1.75} = 0.071$, and x, or $1.5 + z$, will be $= 1.5 + 0.071 = 1.571$; which therefore will be the ſecond near value of the greater root of the equation $5x - x^3 = 4$.

Q. E. I.

Art. LXXXVIII. We will now ſubſtitute this number, 1.571, inſtead of x in the compound quantity $5x - x^3$, in order to diſcover whether the reſult of this ſubſtitution will be greater, or leſs, than 4, or the abſolute term of the propoſed

poſed equation $5x - x^3 = 4$, and conſequently whether 1.571 is leſs, or greater, than the true value of the greater root of that equation.

Now, if x is $= 1.571$, we ſhall have $x^3 = 4.018,682,411$, and $5x\ (= 5 \times 1.571) = 7.855$; and conſequently $5x - x^3$ will be $= 7.855 - 4.018,682,411 = 3.836,317,589$, which is leſs than 4, or the abſolute term of the equation $5x - x^3 = 4$. Therefore 1.571 will be greater than the true value of the greater root of the ſaid equation.

Q. E. I.

In order therefore to obtain a more exact value of this greater root, we will now ſuppoſe it to be equal to $1.571 - z$, and will ſubſtitute this binomial quantity inſtead of x in the propoſed equation $5x - x^3 = 4$, omitting all the terms that involve either the cube, or the ſquare, of z, and will reſolve the new equation obtained by ſuch ſubſtitution.

Now, ſince x is $= 1.571 - z$, we ſhall have

$$x^3 = \overline{1.571-z}^3\ (= \overline{1.571}^3 - 3 \times \overline{1.571}^2 \times z + \&c$$
$$= \overline{1.571}^3 - 3 \times 2.558,041 \times z + \&c$$
$$= \overline{1.571}^3 - 7.674,123 \times z + \&c)$$
$$= 4.018,682,411 - 7.674,123 \times z + \&c,$$

and $5x = 5 \times \overline{1.571-z}\ (= 5 \times 1.571 - 5 \times z) =$
$7.855 - 5z.$

Therefore

Therefore $5x - x^3$ will be $= 7.855 - 5z$
$- \sqrt{4.018,682,411 - 7.674,123 \times z} =$
$\left\{\begin{array}{r} 7.855,000,000 - 5.000,000 \times z \\ - 4.018,682,411 + 7.674,123 \times z \end{array}\right\}$
$= 3.836,317,589 + 2.674,123 \times z.$

But $5x - x^3$ is $= 4$.

Therefore $3.836,317,589 + 2.674,123 \times z$ will alſo be $= 4$. And conſequently $2.674,123 \times z$ will be $(= 4 - 3.836,317,589) = 0.163,682,411$, and z will be $= \frac{0.163,682,411}{2.674,123} = 0.061$. Therefore x, or $1.571 - z$, will be $= 1.571 - 0.061 = 1.510$; and conſequently 1.510 will be the third near value of the greater root of the equation $5x - x^3 = 4$.

Q. E. I.

Art. LXXXIX. Now let 1.510 be ſubſtituted inſtead of x in the compound quantity $5x - x^3$, in order to diſcover whether the reſult will be greater or leſs than the number 4, or the abſolute term of the equation $5x - x^3 = 4$, and conſequently whether 1.510 is leſs, or greater, than the true value of the greater root of the ſaid equation.

Now, if x is $= 1.510$, or 1.51, we ſhall have $x^3 = 3.442,951$, and $5x\ (= 5 \times 1.51) = 7.55$, and conſequently $5x - x^3\ (= 7.55 - 3.442,951) = 4.107,049$; which is greater than 4, or the abſolute term of the equation

5

$5x - x^3 = 4$. Therefore 1.51 is leſs than the true value of the greater root of that equation.

Q. E. I.

In order therefore to obtain a more exact value of this greater root of the equation $5x - x^3 = 4$, let us ſuppoſe x to be $= 1.51 + z$, and proceed as before.

Then, ſince x is $= 1.51 + z$, we ſhall have

$$\begin{aligned} x^3 &= \overline{1.51 + z}^3 \; (= \overline{1.51}^3 + 3 \times \overline{1.51}^2 \times z + \&c \\ &= \overline{1.51}^3 + 3 \times 2.2801 \times z + \&c \\ &= \overline{1.51}^3 + 6.8403 \times z + \&c) \\ &= 3.442{,}951 + 6.8403 \times z + \&c, \end{aligned}$$

$$\begin{aligned} \text{and } 5x &= 5 \times \overline{1.51 + z} \; (= 5 \times 1.51 + 5 \times z) \\ &= 7.55 + 5z. \end{aligned}$$

Therefore $5x - x^3$ will be

$$= \left\{ \begin{array}{r} 7.550{,}000 + 5.0000 \times z \\ - \; 3.442{,}951 - 6.8403 \times z \end{array} \right\}$$

$$= 4.107{,}049 - 1.8403 \times z.$$

But $5x - x^3$ is $= 4$.

Therefore $4.107{,}049 - 1.8403 \times z$ will alſo be $= 4$. Therefore $4.107{,}049$ will be $= 4 + 1.8403 \times z$, and $1.8403 \times z$ will be $(= 4.107{,}049 - 4) = 0.107{,}049$, and z will be $= \frac{0.107{,}049}{1.8403} = 0.0581$. Therefore x, or $1.51 + z$ will be $= 1.51 + 0.0581 = 1.5681$; and conſequently

quently 1.5681 will be a fourth near value of the greater root of the propoſed equation $5x - x^3 = 4$.

Q. E. I.

Art. XC. Now let this number 1.5681 be ſubſtituted inſtead of x in the compound quantity $5x - x^3$. And we ſhall have $5x$ $(= 5 \times 1.5681) = 7.8405$, and x^3 $(= \overline{1.5681}^3) = 3.855,860,066,241$, and conſequently $5x - x^3$ $(= 7.8405 - 3.855,860,066,241) = 3.984,639,933,759$, which is ſomewhat leſs than 4, or the abſolute term of the equation $5x - x^3 = 4$. Therefore 1.5681 is ſomewhat greater than the true value of the greater root of that equation.

In order therefore to obtain a more exact value of this greater root of the equation $5x - x^3 = 4$, let us ſuppoſe x to be $= 1.5681 - z$.

Then we ſhall have $x^3 = \overline{1.5681 - z}^3$ $(= \overline{1.5681}^3 - 3 \times \overline{1.5681}^2 \times z + \&c = \overline{1.5681}^3 - 3 \times 2.458,937,61 \times z + \&c) = 3.855,860,066,241 - 7.376,812,83 \times z + \&c$, and $5x = 5 \times \overline{1.5681 - z}$ $(= 5 \times 1.5681 - 5 \times z) = 7.8405 - 5z$.

Therefore $5x - x^3$ will be $= 7.8405 - 5z - \overline{3.855,860,066,241 - 7.376,812,83 \times z + \&c}$

$$= \left\{ \begin{array}{l} 7.840,500,000,000 - 5.000,000,00 \times z \\ - 3.855,860,066,241 + 7.376,812,83 \times z \end{array} \right\}$$

$= 3.984,639,933,759 + 2.376,812,83 \times z$.

But

But $5x - x^3$ is $= 4$.

Therefore 3.984,639,933,759 + 2.376,812,83 × z will alſo be = 4. And conſequently 2.376,812,83 × z will be ($= 4 - 3.984,639,933,759$) = 0.015,360,066,241, and z will be $= \frac{0.015,360,066,241}{2.376,812,83} = 0.006,46$. Therefore x, or $1,5681 - z$, will be $= 1.5681 - 0.006,46 = 1.561,64$, and conſequently 1.561,64 will be a fifth near value of the greater root of the propoſed equation $5x - x^3 = 4$.

Q. E. I.

Art. XCI. Now let this number 1.561,64 be ſubſtituted inſtead of x in the compound quantity $5x - x^3$. And we ſhall have $5x$ ($= 5 \times 1.561,64$) $= 7.808,20$, and x^3 ($= \overline{1.561,64}^3$) $= 3.808,401,903,738,944$. Therefore $5x - x^3$ will be ($= 7.808,20 - 3.808,401,903,738,944$) $= 3.999,798,096,261,056$, which is ſomewhat leſs than 4, or the abſolute term of the equation $5x - x^3 = 4$. Therefore 1.561,64 muſt be ſomewhat greater than the true value of the greater root of the ſaid equation; but the difference can be but ſmall.

In order therefore to obtain the value of the greater root of this equation to a greater degree of exactneſs, we will continue the proceſs of approximation one ſtep further, and will ſuppoſe x to be $= 1.561,64 - z$, and ſubſtitute this binomial quantity inſtead of it in the equation $5x - x^3 = 4$, omitting (as before,) all the terms that involve either the cube, or the ſquare, of z, and will then reſolve the new equation reſulting from the ſaid ſubſtitution.

Now,

Now, since x is $= 1.561,64 - z$, we shall have

$5x = 5 \times \overline{1.561,64 - z} \; (= 5 \times 1.561,64 - 5 \times z)$
$= 7.808,20 - 5z,$

and $x^3 = \overline{1.561,64 - z}^3 \; (= \overline{1.561,64}^3 - 3 \times \overline{1.561,64}^2 \times z + \&c$
$= \overline{1.561,64}^3 - 3 \times 2.438,719,489,6 \times z + \&c$
$= \overline{1.561,64}^3 - 7.316,158,468,8 \times z + \&c)$
$= 3.808,401,903,738,944 - 7.316,158,468,8 \times z + \&c.$

Therefore $5x - x^3$ will be $= 7.808,20 - 5z$
$- \overline{3.808,401,903,738,944 - 7.316,158,468,8 \times z + \&c}$
$= \left\{ \begin{array}{l} 7.808,200,000,000,000 - 5.000,000,000,0 \times z \\ -3.808,401,903,738,944 + 7.316,158,468,8 \times z - \&c \end{array} \right\}$
$= 3.999,798,096,261,056 + 2.316,158,468,8 \times z - \&c.$

But $5x - x^3$ is $= 4$.

Therefore $3.999,798,096,261,056 + 2.316,158,468,8 \times z$ will also be $= 4$. And consequently $2.316,158,468,8 \times z$ will be $(= 4 - 3.999,798,096,261,056) = 0.000,201,903,738,944$, and z will be $= \frac{0.000,201,903,738,944}{2.316,158,468,8}$ $= 0.000,087,171$. Therefore x, or $1.561,64 - z$, will be $= 1.561,64 - 0.000,087,171 = 1.561,552,829$; and consequently this number $1.561,552,829$ will be a sixth near value of the greater root of the proposed equation $5x - x^3 = 4$.

Q. E. I.

Art.

Art. XCII. Of this number 1.561,552,829, the first eight figures 1.561,552,8, are exact; the more accurate value of the greater root of the equation $5x - x^3 = 4$ being 1.561,552,812,8 &c, or $\frac{3.123,105,625,6 \text{ \&c}}{2}$, or $\frac{4.123,105,625,6 \text{ \&c} - 1}{2}$, or $\frac{\sqrt{17} - 1}{2}$. For if x is $= \frac{\sqrt{17} - 1}{2}$, we shall have $5x = 5 \times \left(\frac{\sqrt{17} - 1}{2}\right) = \frac{5 \times \sqrt{17} \quad 5 \times 1}{2} = \frac{5\sqrt{17} - 5}{2}$, and $x^3 = \left(\frac{\sqrt{17} - 1}{2}\right)^3 \left(= \frac{\sqrt{17} - 1^3}{2^3} = \frac{\sqrt{17} - 1^3}{8} = \right.$

$\frac{17 \times \sqrt{17} - 3 \times 17 \times 1 + 3 \times \sqrt{17} \times 1 - 1}{8} =$

$\frac{17 \times \sqrt{17} - 51 + 3 \times \sqrt{17} - 1}{8} = \frac{20 \times \sqrt{17} - 52}{8}$

$= \frac{5 \times 4 \times \sqrt{17} - 13 \times 4}{2 \times 4}) = \frac{5 \times \sqrt{17} - 13}{2}$.

Therefore $5x - x^3$ will be $= \frac{5\sqrt{17} - 5}{2} - \frac{5\sqrt{17} - 13}{2}$

$= \frac{5\sqrt{17} - 5 - 5\sqrt{17} + 13}{2} = \frac{13 - 5}{2} = \frac{8}{2} = 4.$

Therefore $\frac{\sqrt{17} - 1}{2}$ will be exactly equal to the greater value of x in the equation $5x - x^3 = 4$, or will be the greater root of that equation.

Q. E. D.

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A SCHOLIUM.

Art. XCIII. THIS equation $5x - x^3 = 4$ has been uncommonly difficult to reſolve, having required no leſs than five repetitions of Mr. Raphſon's proceſs of approximation to the value of it's greater root after having fixed upon 1.5 for it's firſt near value, before we could obtain the value of the ſaid greater root exact to eight places of figures. And it is remarkable that in reſolving the oppoſite equation $x^3 - 5x = 4$ by the ſame method of approximation in Art. 41, 42, 43, 44, &c, three repetitions of the proceſs of approximation were found to be ſufficient to obtain the value of the root of that equation exact to eleven places of figures. It would therefore have been eaſier for us, when the preſent equation $5x - x^3 = 4$ was propoſed to us to be reſolved, to have firſt reſolved the oppoſite equation $x^3 - 5x = 4$ to the intended degree of exactneſs, and then, calling the root of that equation z, to have derived the greater root of the preſent equation $5x - x^3 = 4$ from it by computing the expreſſion $\frac{z + \sqrt{4b - 3zz}}{2}$, which, (by Art. 76) is equal to t, or the greater root of the equation $bx - x^3 = c$, or $5x - x^3 = 4$. For, if this courſe had been taken to obtain the value of t, we ſhould have had $z = 2.561,552,812$, &c, and conſequently $zz\ (= \overline{2.561,552,812}^2) = 6.561,552,808,665,107,344$, and $3zz\ (= 3 \times 6.561,552,808,665,107,344) = 19.684,658,425,995,322,032$, and $4b - 3zz\ (= 4 \times 5 - 3zz = 20 - 19.684,658,425,995,322,032) = 0.315,$

341,574,004,677,968, and consequently $\sqrt{4b - 3zz}$ (= $\sqrt{0.315,341,574,004,677,968}$) = 0.561,552,823, and $z + \sqrt{4b - 3zz}$ (= 2.561,552,812 + 0.561,552,812) = 3.123,105,635, and $\frac{z + \sqrt{4b - 3zz}}{2}$ (= $\frac{3.123,105,635}{2}$) = 1.561,552,817, &c; that is, t, or the greater root of the propoſed equation $3x - x^3 = 4$, would have been found to be = 1.561,552,817, &c; of which number the nine firſt figures, 1.561,552,81, are exact. And this computation of the expreſſion $\frac{z + \sqrt{4b - 3zz}}{2}$ would have been leſs troubleſome than the performance of the fourth and fifth proceſs of approximation which were the operations, correſponding to this computation, that were neceſſary to obtain the value of this root to the eight figures 1.561,552,8, in the foregoing articles. It may therefore ſometimes happen, when an equation of the form $bx - x^3 = c$ is propoſed to us to be reſolved, that it will be eaſier to reſolve it in an indirect manner, by firſt reſolving the oppoſite equation $x^3 - bx = c$, or $x^3 - bx = -c$, and then computing the expreſſion $\frac{z + \sqrt{4b - 3zz}}{2}$, (as we have juſt now done,) in order to find the value of t, or it's greater root, than to reſolve it directly by the method deſcribed in Problem 4: and I am inclined to think that this will often be the eaſier. But this is a matter that muſt be left to the diſcretion of the Algebraiſt himſelf, who has occaſion to reſolve any equations of this kind, in every particular equation that occurs to him; as I believe no general rules can ſafely be given to determine

when

when the direct method of resolving them, described in Problem iv, is to be preferred to the indirect method of resolving them, by first resolving the opposite equation $x^3 - bx = c$, and then computing the expression $\frac{z + \sqrt{4b - 3zz}}{2}$, and when it is most convenient to resort at once to the said indirect method.

Art. XCIV. As to the lesser root of the equation $5x - x^3 = 4$, it is easy to see that it is 1. For, if x is $= 1$, we shall have x^3 also $= 1$, and $5x\ (= 5 \times 1) = 5$, and consequently $5x - x^3\ (= 5 - 1) = 4$. Therefore 1 must be equal to x, or must be a root of the equation $5x - x^3 = 4$.

Q. E. D.

And we may likewise derive this lesser root of the equation $5x - x^3 = 4$ from it's greater root t, which has been shewn in Art. 92, to be $= \frac{\sqrt{17} - 1}{2}$, by computing the expression $\frac{\sqrt{4b - 3tt} - t}{2}$, to which it has been shewn in Art. 73, that the lesser root of the said equation must always be equal. This computation may be made as follows:

If t is $= \frac{\sqrt{17} - 1}{2}$, tt will be $= \left(\frac{\sqrt{17} - 1}{2}\right)^2$ $\left(= \frac{\sqrt{17-1}^2}{2^2} = \frac{\sqrt{17-1}^2}{4} = \frac{17 - 2\sqrt{17} + 1}{4} = \frac{18 - 2\sqrt{17}}{4}\right)$

=

$= \frac{9-\sqrt{17}}{2}$, and consequently $3tt$ will be $= 3 \times \frac{9-\sqrt{17}}{2} \left(= \frac{3 \times 9 - 3 \times \sqrt{17}}{2}\right) = \frac{27-3\sqrt{17}}{2}$, and $4b - 3tt$ will be $= 4 \times 5 - \frac{27-3\sqrt{17}}{2} \left(= 20 - \frac{27-3\sqrt{17}}{2} = \frac{40}{2} - \frac{27-3\sqrt{17}}{2} = \frac{40-27+3\sqrt{17}}{2} = \frac{13+3\sqrt{17}}{2} = \frac{26+6\sqrt{17}}{4} = \frac{9+2\times 3\times\sqrt{17}+17}{4}\right) = \left(\frac{3+\sqrt{17}}{2}\right)^2$. Therefore $\sqrt{4b-3tt}$ will be $= \frac{3+\sqrt{17}}{2}$, and consequently $\sqrt{4b-3tt} - t$ will be $= \frac{3+\sqrt{17}}{2} - \frac{\sqrt{17}-1}{2} \left(= \frac{3+\sqrt{17}-\sqrt{17}+1}{2} = \frac{3+1}{2} = \frac{4}{2}\right) = 2$, and $\frac{\sqrt{4b-3tt}-t}{2}$ will be $= \frac{2}{2} = 1$. Therefore x, or the leſſer root of the equation $5x - x^3 = 4$, will be $= 1$.

Q. E. I.

EXAMPLE III.

Art. XCV. LET it be required to find the two roots of the equation $63x - x^3 = 162$.

Here

Here b is $= 63$, and c is $= 162$; and conſequently $\frac{c}{2}$ is $= 81$, and $\frac{cc}{4}$ is $(= \overline{81}^2) = 6561$; and $\frac{b}{3}$ is $(= \frac{63}{3}) = 21$, and $\frac{b^3}{27}$ is $(= \overline{21}^3) = 9261$, which is greater than 6561, or $\frac{cc}{4}$. Therefore cc will be leſs than $\frac{4b^3}{27}$, and c will be leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$. And conſequently the equation $63x - x^3 = 162$ is a poſſible equation, and will have two roots, one of which will be leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, (or $\sqrt{\frac{b}{3}}$, or $\sqrt{21}$,) or 4.582, &c, and the other will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, or 4.582, &c, but leſs than $\sqrt{b}$, (or $\sqrt{63}$,) or 7.937, &c. This greater root may be inveſtigated in the following manner:

Art. XCVI. When x is $= \frac{\sqrt{b}}{\sqrt{3}}$, or 4.582, &c, the compound quantity $bx - x^3$, or $63x - x^3$, will be $= \frac{2b\sqrt{b}}{3\sqrt{3}}$ $(= \frac{2b}{3} \times \frac{\sqrt{b}}{\sqrt{3}} = \frac{2b}{3} \times 4.582$, &c $= 2 \times \frac{b}{3} \times 4.582$, &c $= 2 \times 21 \times 4.582$, &c $= 42 \times 4.582$, &c) $= 192.444$, &c; and, when x is $= \sqrt{b}$, (or $\sqrt{63}$,) or 7.937, &c, the ſame compound quantity will be $= 0$. But in the preſent caſe the ſaid compound quantity is $= 162$, which approaches much nearer to 192.444, &c, which is it's magnitude when x is $= 4.582$, &c, than to 0, which is it's magnitude when x is $= 7.937$, &c. Therefore it ſeems reaſonable to ſuppoſe that the greater

value of x in the equation $63x - x^3 = 162$ will be nearer to it's lower limit 4.582, &c, than to it's higher limit 7.937, &c, and consequently that it will be less than half the sum of the said two limits, or than $\frac{4.582 \text{ \&c} + 7.937, \text{ \&c}}{2}$, or than $\frac{12.519, \text{ \&c}}{2}$, or than 6.259, &c. We will therefore conjecture that it is equal, or nearly equal, to 6, and will substitute this number 6 in it's stead in the compound quantity $63x - x^3$, in order to discover whether the result of the said substitution will be equal to, or greater than, or less than, 162, or the absolute term of the equation $63x - x^3 = 162$, and consequently whether the said number 6 is equal to, less than, or greater than, the true value of x, or the greater root of the said equation.

Now, if x is $= 6$, we shall have $x^3 (= 6^3) = 216$, and $63x (= 63 \times 6) = 378$, and consequently $63x - x^3 (= 378 - 216) = 162$. Therefore 6 will be accurately equal to x, or the greater root of the proposed equation $63x - x^3 = 162$.

Q. E. I.

Art. XCVII. Having thus found 6 to be the greater root of the proposed equation $63x - x^3 = 162$, we will derive the value of the lesser root from it by computing the expression $\frac{\sqrt{4b - 3tt} - t}{2}$, in which t stands for 6, or the greater root of the said equation which has been already found.

Now, since t is $= 6$, we shall have $tt = 36$, and consequently

quently $3tt$ ($= 3 \times 36$) $= 108$, and $4b - 3tt = 4b - 108 = 4 \times 63 - 108$ ($= 252 - 108$) $= 144$, and $\sqrt{4b - 3tt}$ ($= \sqrt{144}$) $= 12$, and $\sqrt{4b - 3tt} - t$ ($= 12 - 6$) $= 6$, and $\frac{\sqrt{4b - 3tt} - t}{2}$ ($= \frac{6}{2}$) $= 3$. Therefore 3 will be the leſſer root of the propoſed equation $63x - x^3 = 162$.

Q. E. I.

And we ſhall find, upon trial, that 3 will anſwer the conditions of the equation. For, if x be $= 3$, we ſhall have $x^3 = 27$, and $63x$ ($= 63 \times 3$) $= 189$, and conſequently $63x - x^3$ ($= 189 - 27$) $= 162$, as it ought to be. Therefore 3 is the leſſer root of the propoſed equation $63x - x^3 = 162$.

Q. E. D.

EXAMPLE IV.

Art. XCVIII. LET it be required to find the two roots of the cubick equation $15x - x^3 = 4$.

Here b is $= 15$, and c is $= 4$; and conſequently $\frac{c}{2}$ is $(= \frac{4}{2}) = 2$, and $\frac{cc}{4}$ is $(= \overline{2}^2) = 4$, and $\frac{b}{3}$ is $(= \frac{15}{3}) = 5$, and $\frac{b^3}{27}$ is $(= \overline{5}^3) = 125$, which is greater than 4, or $\frac{cc}{4}$.

$\frac{cc}{4}$. Therefore cc will be leſs than $\frac{4b^3}{27}$, and c will be leſs than $\frac{2b\sqrt{b}}{3\sqrt{3}}$; and conſequently the equation $15x - x^3 = 4$ will be a poſſible equation, and will have two roots, one of which will be leſs than $\frac{\sqrt{b}}{\sqrt{3}}$, (or $\sqrt{\left[\frac{b}{3}\right.}$, or $\sqrt{5}$,) or 2.236, &c, and the other will be greater than $\frac{\sqrt{b}}{\sqrt{3}}$, or 2.236, &c, but leſs than $\sqrt{b}$, or $\sqrt{15}$, or 3.872, &c. This greater root may be inveſtigated as follows:

Art. XCIX. When x is $= \frac{\sqrt{b}}{\sqrt{3}}$, or 2.236, &c, the compound quantity $bx - x^3$, or $15x - x^3$, will be $= \frac{2b\sqrt{b}}{3\sqrt{3}}$ $(= \frac{2b}{3} \times \frac{\sqrt{b}}{\sqrt{3}} = \frac{2b}{3} \times 2.236, \&c = \frac{2 \times 15}{3} \times 2.236, \&c = 2 \times 5 \times 2.236, \&c = 10 \times 2.236, \&c) =$ 22.36 &c; and, when x is $= \sqrt{b}$, or 3.872, &c, the ſame compound quantity will be $= 0$. But in the preſent caſe the ſaid compound quantity is equal to 4, which approaches much nearer to 0, or it's magnitude when the greater value of x is equal to it's higher limit 3.872, &c, than to 22.36 &c, or it's magnitude when the ſaid value of x is equal to it's lower limit, 2.236, &c. Therefore it ſeems reaſonable to ſuppoſe that the ſaid greater value of x will be nearer to it's higher limit, 3.872, &c, than to it's lower limit, 2.236, &c, and conſequently that it will be greater than half the ſum of the ſaid two limits, or than the fraction $\frac{2.236\ \&c + 3.872,\ \&c}{2}$, or than $\frac{6.108}{2}$, or than 3.054. We will therefore conjecture that it is nearly equal

equal to 3.6, and will ſubſtitute 3.6 inſtead of x in the compound quantity $15x - x^3$, in order to diſcover whether the reſult of the ſaid ſubſtitution will be greater than, or leſs than, 4, or the abſolute term of the equation $15x - x^3 = 4$, and conſequently whether the ſaid number 3.6 is leſs than, or greater than, the true value of the greater root of the ſaid equation.

Now, if x is $= 3.6$, we ſhall have $x^3 (= \overline{3.6}|^3) = 46.656$, and $15x (= 15 \times 3.6) = 54.0$. Therefore $15x - x^3$ will be $= 54.000 - 46.656 = 7.344$; which is conſiderably greater than 4, or the abſolute term of the equation $15x - x^3 = 4$. Therefore 3.6 muſt be leſs than the true value of the greater root of that equation; and their difference will, probably, be conſiderable.

We will therefore make a ſecond conjecture, and ſuppoſe x to be $= 3.7$.

Then we ſhall have $x^3 (= \overline{3.7}|^3) = 50.653$, and $15x (= 15 \times 3.7) = 55.5$. Therefore $15x - x^3$ will be $(= 55.500 - 50.653) = 4.847$; which is ſtill ſomewhat greater than 4, or the abſolute term of the propoſed equation $15x - x^3 = 4$. Therefore 3.7 muſt be ſomewhat leſs than the true value of the greater root of the ſaid equation. It ſeems, however, to come near enough to the ſaid true value to be made the baſis of a further approximation to it according to the directions of Mr. Raphſon's method.

Art. C. We will therefore ſuppoſe x to be $=$ 3.7

$3.7 + z$, and will ſubſtitute this binomial quantity inſtead of x in the propoſed equation $15x - x^3 = 4$, omitting, (as in the former examples,) all the terms that involve in them either the cube, or the ſquare, of z, and will then reſolve the new equation reſulting from ſuch ſubſtitution.

Now, ſince x is $= 3.7 + z$, we ſhall have $x^3 = \overline{3.7 + z}^3$ $(= \overline{3.7}^3 + 3 \times \overline{3.7}^2 \times z + \&c = \overline{3.7}^3 + 3 \times 13.69 \times z + \&c = \overline{3.7}^3 + 41.07 \times z + \&c) = 50.653 + 41.07 \times z + \&c$,
and $15x = 15 \times \overline{3.7 + z}$ $(= 15 \times 3.7 + 15 \times z (= 55.5 + 15z$, and conſequently $15x - x^3$ $(= 55.5 + 15z - 50.653 - 41.07 \times z) = 4.847 - 26.07 \times z$.

But $15x - x^3$ is $= 4$.

Therefore $4.847 - 26.07 \times z$ will alſo be $= 4$. And conſequently 4.847 will be $= 4 + 26.07 \times z$, and $26.07 \times z$ will be $(= 4.847 - 4) = 0.847$, and z will be $= \frac{0.847}{26.07} = 0.032$. Therefore x, or $3.7 + z$, will be $(= 3.7 + 0.032) = 3.732$; and conſequently 3.732 will be a ſecond near value of the greater root of the propoſed equation $15x - x^3 = 4$.

Q. E. I.

Art. CI. Now let this number, 3.732, be ſubſtituted inſtead of x in the compound quantity $15x - x^3$, in order to diſcover, whether the reſult of the ſaid ſubſtitution will be greater, or leſs, than 4, or the abſolute term of the equation $15x - x^3 = 4$, and conſequently whether 3.732 will

will be leſs, or greater, than the true value of the greater root of the ſaid equation.

Now, if x is $= 3.732$, we ſhall have $x^3 (= \overline{3.732}^3) = 51.978,639,168$, and $15x (= 15 \times 3.732) = 55.980$, and conſequently $15x - x^3 (= 55.980 - 51.978,639,168) = 4.001,360,832$; which is a little greater than 4, or the abſolute term of the equation $15x - x^3 = 4$. Therefore 3.732 muſt be a little leſs than the true value of the greater root of the ſaid equation.

In order therefore to obtain a more accurate value of the greater root of this equation, we will ſuppoſe it to be equal to $3.732 + z$, and will ſubſtitute this binomial quantity inſtead of it in the ſaid equation, omitting, as before, all the terms that involve either the cube, or the ſquare, of z, and will then reſolve the new equation reſulting from ſuch ſubſtitution.

Now, if x is $= 3.732 + z$, we ſhall have

$$x^3 = \overline{3.732 + z}^3 (= \overline{3.732}^3 + 3 \times \overline{3.732}^2 \times z + \&c$$
$$= \overline{3.732}^3 + 3 \times 13.927,824 \times z + \&c =$$
$$\overline{3.732}^3 + 41.783,472 \times z + \&c) =$$
$$51.978,639,168 + 41.783,472 \times z + \&c,$$

and $15x = 15 \times \overline{3.732 + z} (= 15 \times 3.732 + 15 \times z)$
$= 55.980 + 15z.$

Therefore $15x - x^3$ will be $=$

$$\left\{\begin{array}{l} 55.980,000,000 + 15.000,000 \times z \\ -\ 51.978,639,168 - 41.783,472 \times z \end{array}\right\} =$$
$$4.001,360,832 - 26.783,472 \times z.$$

But

But $15x - x^3$ is $= 4$.

Therefore $4.001,360,832 - 26.783,472 \times z$ will alſo be $= 4$. And conſequently $4.001,360,832$ will be $= 4 + 26.783,472 \times z$, and $26.783,472 \times z$ will be $(= 4.001,360,832 - 4) = 0.001,360,832$, and z will be $= \frac{0.001,360,832}{26.783,472} = 0.000,050,808$. Therefore x, or $3.732 + z$, will be $(= 3.732 + 0.000,050,808) = 3.732,050,808$; and conſequently $3.732,050,808$ will be a third near value of the greater root of the propoſed equation $15x - x^3 = 4$.

Q. E. I.

Art. CII. This value of the greater root of the equation $15x - x^3 = 4$ is exact in the firſt nine figures, $3.732,050,80$; the more accurate value of it being $3.732,050,807,568,877,3$ &c, or $2 + 1.732,050,807,568,877,3$ &c, or $2 + \sqrt{3}$. This will appear by ſubſtituting $2 + \sqrt{3}$ inſtead of x in the compound quantity $15x - x^3$. For, if this be done, the reſult will be equal to 4, or the abſolute term of the propoſed equation $15x - x^3 = 4$. For x^3 will be $= \overline{2+\sqrt{3}}^3$ $(= \overline{2}^3 + 3 \times \overline{2}^2 \times \sqrt{3} + 3 \times 2 \times 3 + 3\sqrt{3} = 8 + 3 \times 4 \times \sqrt{3} + 18 + 3 \times \sqrt{3} = 8 + 12 \times \sqrt{3} + 18 + 3 \times \sqrt{3}) = 26 + 15 \times \sqrt{3}$, and $15x$ will be $= 15 \times \overline{2 + \sqrt{3}}$ $(= 15 \times 2 + 15 \times \sqrt{3}) = 30 + 15 \times \sqrt{3}$; and conſequently $15x - x^3$ will be $= 30 + 15 \times \sqrt{3} - 26 - 15 \times \sqrt{3} (= 30 - 26) = 4$. Therefore $2 + \sqrt{3}$ muſt be equal to one of the roots of the propoſed equation $15x - x^3 = 4$.

Q. E. D.

Art.

Art. CIII. Having thus found $2 + \sqrt{3}$ to be the greater root of the propoſed equation $15x - x^3 = 4$, we will derive the value of the leſſer root of it from the greater root by computing the expreſſion $\frac{\sqrt{4b - 3tt} - t}{2}$, in which t ſtands for $2 + \sqrt{3}$, or the greater root of the ſaid equation, which we have already found.

Now, ſince t is $= 2 + \sqrt{3}$, we ſhall have $tt = \overline{2 + \sqrt{3}}^2$ $(= \overline{2}^2 + 2 \times 2 \times \sqrt{3} + 3 = 4 + 4\sqrt{3} + 3) = 7 + 4\sqrt{3}$, and conſequently $3tt$ $(= 3 \times \overline{7 + 4\sqrt{3}} = 3 \times 7 + 3 \times 4\sqrt{3}) = 21 + 12\sqrt{3}$, and $4b - 3tt = 4 \times 15 - \overline{21 + 12\sqrt{3}}$ $(= 4 \times 15 - 21 - 12\sqrt{3} = 60 - 21 - 12\sqrt{3} = 39 - 12\sqrt{3} = 36 - 12\sqrt{3} + 3 = \overline{6}^2 - 2 \times 6 \times \sqrt{3} + 3) = \overline{6 - \sqrt{3}}^2$. Therefore $\sqrt{4b - 3tt}$ will be $= 6 - \sqrt{3}$, and $\sqrt{4b - 3tt} - t$ will be $= 6 - \sqrt{3} - \overline{2 + \sqrt{3}}$ $(= 6 - \sqrt{3} - 2 - \sqrt{3} = 6 - 2 - 2\sqrt{3} = 4 - 2\sqrt{3}) = 2 \times \overline{2 - \sqrt{3}}$, and $\frac{\sqrt{4b - 3tt} - t}{2}$ will be $\left(= \frac{2 \times \overline{2 - \sqrt{3}}}{2}\right) = 2 - \sqrt{3}$. Therefore the leſſer root of the propoſed equation $15x - x^3 = 4$ is $= 2 - \sqrt{3}$.

Q. E. I.

And that this quantity $2 - \sqrt{3}$ is equal to the leſſer root of the propoſed equation $15x - x^3 = 4$ will alſo appear by ſubſtituting it inſtead of x in the compound quantity $15x - x^3$. For the value of the ſaid compound quantity ariſing from this ſubſtitution will be found to be 4, or the abſolute term of the ſaid equation. For, if x is $= 2 - \sqrt{3}$, we

of theſe equations that are required by Problem iv. And therefore, when our object is to find the roots of an equation of the form $bx - x^3 = c$ exact to eight or more places of figures, I will venture to aſſert that it will be much more convenient to inveſtigate them by the methods deſcribed above in Problems iii and iv than to have recourſe to the triſection of a circular arc for that purpoſe. And I alſo believe that thoſe methods of reſolving equations of this kind will be more convenient, in moſt caſes, than any other methods that can be taken for the purpoſe.

END of the Reſolution of Cubick Equations of the three foregoing Forms, $x^3 + bx = c$, $x^3 - bx = c$, *and* $bx - x^3 = c$.

Art. CV. We have now ſhewn, by the ſolution of the foregoing four Problems, how all the three cubick equations, $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, in which the ſquare of the unknown quantity x is wanting, may be reſolved. And to theſe three forms all the other forms of cubick equations may be reduced, by making certain proper ſubſtitutions in them, by which the term involving the ſquare of the unknown quantity will be expunged: and conſequently all the ſaid other forms of cubick equations may ultimately be reſolved by means of the reſolutions above-explained of equations of theſe three forms. The manner of reducing thoſe other forms of cubick equations to ſome of the three forms above-mentioned may be explained as follows:

OF

OF CUBICK EQUATIONS CONTAINING THE SQUARE OF THE UNKNOWN QUANTITY x, BUT WANTING IT'S SIMPLE POWER, OR x ITSELF.

Art. CVI. THE equations that have in them the ſquare of the unknown quantity x, but are without it's ſimple power, or x itſelf, are only the three following; to wit,

$$\text{1ſt,} \quad x^3 + ax^2 = c,$$
$$\text{2dly,} \quad x^3 - ax^2 = c,$$
$$\text{and 3dly,} \quad ax^2 - x^3 = c.$$

For ax^2 may either, in the 1ſt place, be added to x^3, (in which caſe ax^2 may be either leſs than, or equal to, or greater than, x^3,) and their ſum may be declared to be equal to the known quantity, or abſolute term, c; and then we ſhall have the firſt of theſe three equations, to wit, $x^3 + ax^2 = c$;

Or, in the 2d place, ax^2 may be ſubtracted from x^3, (in which caſe it muſt be leſs than x^3,) and their difference may be declared to be equal to the known quantity, or abſolute term, c; and then we ſhall have the ſecond of theſe three equations, to wit, $x^3 - ax^2 = c$;

Or, 3dly, x^3 may be ſubtracted from ax^2, (in which caſe it muſt be leſs than ax^2, or ax^2 muſt be greater than x^3,)

x^3,) and their difference may be declared to be equal to the known quantity, or abſolute term, c; and then we ſhall have the laſt of theſe three equations, to wit, $ax^2 - x^3 = c$. No other combinations of the two terms x^3 and ax^2 by addition or ſubtraction are poſſible.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $x^3 + ax^2 = c$.

Art. CVII. TO exterminate the ſecond term, ax^2, of the equation $x^3 + ax^2 = c$, let us ſuppoſe y to be a quantity that is greater than $\frac{a}{3}$, or the third part of the coefficient of x^2 in the equation $x^3 + ax^2 = c$, and that exceeds it by a quantity that is equal to x, ſo that $y - \frac{a}{3}$ ſhall be equal to x. This ſuppoſition is evidently poſſible; becauſe, whatever be the magnitude of x, it will always be poſſible for ſome quantity greater than $\frac{a}{3}$ to exiſt, which ſhall exceed $\frac{a}{3}$ by an exceſs that is equal to it. Now let this quantity $y - \frac{a}{3}$ (which is equal to x,) be ſubſtituted inſtead of x in the equation $x^3 + ax^2 = c$; and we ſhall have $x^3 = \overline{y - \frac{a}{3}}^3 \left(= y^3 - 3 \times yy \times \frac{a}{3} + 3 \times y \times \frac{aa}{9} - \frac{a^3}{27}\right) = y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27}$, and

and $x^2 = \overline{y - \frac{a}{3}}^2 \left(= yy - 2y \times \frac{a}{3} + \frac{aa}{9}\right) = yy - \frac{2ay}{3} + \frac{aa}{9}$, and conſequently $ax^2 \left(= a \times \overline{yy - \frac{2ay}{3} + \frac{aa}{9}}\right) = ayy - \frac{2aay}{3} + \frac{a^3}{9}$, and $x^3 + ax^2$

$$= \left\{\begin{array}{l} y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27} \\ \quad + ayy - \frac{2aay}{3} + \frac{a^3}{9} \end{array}\right\} =$$

$$y^3 * - \frac{aay}{3} - \frac{a^3}{27} + \frac{a^3}{9} \left\{\begin{array}{l} = y^3 * - \frac{aay}{3} - \frac{a^3}{27} \\ \qquad\qquad + \frac{3a^3}{27} \end{array}\right\}$$

$$= y^3 * - \frac{aay}{3} + \frac{2a^3}{27}.$$

But $x^3 + ax^2$ is $= c$.

Therefore $y^3 * - \frac{aay}{3} + \frac{2a^3}{27}$ will alſo be $= c$; which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore will come under one of the three forms $x^3 + bx = c$, and $x^3 - bx = c$, and $bx - x^3 = c$, and therefore may be reſolved by means of one of the four Problems above-explained.

If c is greater than $\frac{2a^3}{27}$, let $\frac{2a^3}{27}$ be ſubtracted from both ſides of the laſt equation $y^3 * - \frac{aay}{3} + \frac{2a^3}{27} = c$; and

we ſhall have $y^3 - \frac{aay}{3} = c - \frac{2a^3}{27}$, or $y^3 - \frac{aay}{3} = \frac{27c - 2a^3}{27}$; which comes under the ſecond of the three foregoing forms, to wit, the form $x^3 - bx = c$, and therefore may be reſolved either by Problem ii. or by Problem iii. And, if c is leſs than $\frac{2a^3}{27}$, we muſt add $\frac{aay}{3}$ to both ſides of the equation $y^3 * - \frac{aay}{3} + \frac{2a^3}{27} = c$; and we ſhall then have $y^3 + \frac{2}{27} = c + \frac{aay}{3}$, and (ſubtracting y^3 from both ſides,) $\frac{2a^3}{27} = c + \frac{aay}{3} - y^3$, and, laſtly, (ſubtracting c from both ſides,) $\frac{2a^3}{27} - c = \frac{aay}{3} - y^3$, or $\frac{aay}{3} - y^3 = \frac{2a^3}{27} - c$, or $\frac{aay}{3} - y^3 = \frac{2a^3 - 27c}{27}$; which comes under the 3d of the three foregoing forms, to wit, the form $bx - x^3 = c$, and therefore may be reſolved by Problem iv.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $x^3 - ax^2 = c$.

Art. CVIII. IN this equation x is always greater than a, and conſequently, *à fortiori*, greater than $\frac{a}{3}$. Let it's

exceſs

excefs above $\frac{a}{3}$ be called y; that is, let y be $= x - \frac{a}{3}$. Then will x be $= y + \frac{a}{3}$, and conſequently x^3 will be $= \overline{y + \frac{a}{3}}^3 \left(= y^3 + 3yy \times \frac{a}{3} + 3yy \times \frac{aa}{9} + \frac{a^3}{27}\right)$ $= y^3 + ayy + \frac{aay}{3} + \frac{a^3}{27}$, and x^2 will be $= \overline{y + \frac{a}{3}}^2 \left(= yy + 2y \times \frac{a}{3} + \frac{aa}{9}\right) = yy + \frac{2ay}{3} + \frac{aa}{9}$, and ax^2 will be $\left(= a \times yy + \frac{2ay}{3} + \frac{aa}{9}\right) = ayy + \frac{2a^2y}{3} + \frac{a^3}{9}$.

Therefore $x^3 - ax^2$ will be $= \left\{\begin{array}{l} y^3 + ayy + \frac{aay}{3} + \frac{a^3}{27} \\ \quad - ayy - \frac{2a^2y}{3} - \frac{a^3}{9} \end{array}\right\} =$

$$y^3 * - \frac{a^2y}{3} + \frac{a^3}{27} - \frac{3a^3}{27} \Big\} = y^3 * - \frac{a^2y}{3} - \frac{2a^3}{27}$$

But $x^3 - ax^2$ is $= c$.

Therefore $y^3 * - \frac{a^2y}{3} - \frac{2a^3}{27}$ will alſo be $= c$; and conſequently (adding $\frac{2a^3}{27}$ to both ſides,) $y^3 * - \frac{a^2y}{3}$ will be $= c + \frac{2a^3}{27}$, or $\frac{27c + 2a^3}{27}$; which is an equation of the ſame form with the equation $x^3 - bx = c$, and therefore may be reſolved either by Problem. ii or Problem iii.

OF THE EXTERMINATION OF THE TERM ax^2 FROM THE EQUATION $ax^2 - x^3 = c$.

Art. CIX. THIS equation will not always be poſſible, whatever be the magnitude of the abſolute term c, but only when c is not greater than the greateſt poſſible magnitude to which the compound quantity $ax^2 - x^3$ can increaſe while x increaſes from 0 to a, which is it's greateſt poſſible magnitude. Now the greateſt poſſible magnitude of the compound quantity $ax^2 - x^3$ may be found in the following manner:

Let the quantity x be ſuppoſed to increaſe from 0 to it's greateſt magnitude a by the continual addition of ſome very ſmall quantity, (as, for example, the ten-thouſand millionth part of x,) which we will denote by $\dot{x}$, or the letter x with a point placed over it. Then it is evident that, while x increaſes from x to $x + \dot{x}$, x^2 will increaſe from x^2 to $\overline{x + \dot{x}}^2$, or $x^2 + 2x\dot{x} + \dot{x}\dot{x}$, and x^3 will increaſe from x^3 to $\overline{x + \dot{x}}^3$, or $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, and ax^2 will increaſe from ax^2 to $a \times \overline{x^2 + 2x\dot{x} + \dot{x}\dot{x}}$, or $ax^2 + 2ax\dot{x} + a\dot{x}\dot{x}$; ſo that the increments of ax^2 and x^3 that are contemporary with the increment $\dot{x}$ will be $2ax\dot{x} + a\dot{x}\dot{x}$ and $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, or (neglecting $a\dot{x}\dot{x}$ and $3x\dot{x}^2 + x^3$ on account of their extreme ſmallneſs in compariſon of $2ax\dot{x}$ and $3x^2\dot{x}$,) will be $2ax\dot{x}$ and $3x^2\dot{x}$. Now the increment $2ax\dot{x}$ is to the increment $3x^2\dot{x}$ in the proportion of $2ax$ to $3x^2$, or of $2a$ to $3x$, which, when x is much ſmaller than a, is a very great ratio of majority, and will continue

continue to be a ratio of majority ſo long as $2a$ is greater than $3x$, or $\frac{2a}{3}$ is greater than x. But when x is become equal to $\frac{2a}{3}$, the increment $2axx$ will be equal to the increment $3x^2x$; and, when x is greater than $\frac{2a}{3}$, though leſs than a, the increment $2axx$ will be leſs than the increment $3x^2x$. Therefore, while x is increaſing from 0 to $\frac{2a}{3}$, the increment of ax^2, the firſt and greater member of the compound quantity $ax^2 - x^3$, will be greater than the contemporary increment of x^3, the ſecond and leſſer member of the ſaid compound quantity, which is ſubtracted from it's greater member ax^2; and therefore the ſaid compound quantity muſt, upon the whole, increaſe; but, afterwards, while x is increaſing further from $\frac{2a}{3}$ to a, the increment of ax^2 will be leſs than the contemporary increment of x^3; and therefore the ſaid compound quantity muſt, upon the whole, decreaſe. And conſequently the greateſt magnitude of the ſaid compound quantity $ax^2 - x^3$ will be that which it has when it ceaſes to increaſe and begins to decreaſe, or that which it has when x is $= \frac{2a}{3}$. And this magnitude is $a \times \overline{\frac{2a}{3}}^2 - \overline{\frac{2a}{3}}^3$ $(= a \times \frac{4aa}{9} - \frac{8a^3}{27} = \frac{4a^3}{9} - \frac{8a^3}{27} = \frac{\times 4a^3}{27} - \frac{8a^3}{27} = \frac{12a^3}{27} - \frac{8a^3}{27}) = \frac{4a^3}{27}$. Therefore, while x increaſes from 0 to $\frac{2a}{3}$, the compound quantity $ax^2 - x^3$

will increaſe from o to $\frac{4a^3}{27}$; and, while x increaſes further from $\frac{2a}{3}$ to a, the ſaid compound quantity will decreaſe from $\frac{4a^3}{27}$ to o; and conſequently the greateſt poſſible magnitude of the compound quantity $ax^2 - x^3$ will be $\frac{4a^3}{27}$.

Q. E. I.

Therefore, if c, or the abſolute term of the equation $ax^2 - x^3 = c$, is greater than $\frac{4a^3}{27}$, the ſaid equation will be impoſſible; and, if c is equal to $\frac{4a^3}{27}$, the ſaid equation will have only one root, and that root will be $= \frac{2a}{3}$; and, if c is leſs than $\frac{4a^3}{27}$, the ſaid equation will have two roots, of which the leſſer will be leſs than $\frac{2a}{3}$, and the greater will be greater than $\frac{2a}{3}$, but leſs than a.

Art. CX. Since the greater root of the equation $ax^2 - x^3 = c$ is greater than $\frac{2a}{3}$, it will, *à fortiori*, be greater than $\frac{a}{3}$, and therefore may be ſuppoſed to be equal to

$\frac{a}{3}$

$\frac{a}{3} + y$. Let this ſuppoſition be made, in order to exterminate ax^2 from the equation $ax^2 - x^3 = c$, and to convert the ſaid equation into another equation in which the ſquare of the unknown quantity ſhall be wanting. Then we ſhall have $x^2 = \overline{\frac{a}{3} + y}^2 = \frac{aa}{9} + \frac{2ay}{3} + yy$, and $ax^2 = \frac{a^3}{9} + \frac{2aay}{3} + ayy$, and $x^3 = \overline{\frac{a}{3} + y}^3 (= \frac{a^3}{27} + 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy + y^3) = \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3$,

and conſequently $ax^2 - x^3 =$

$$\left\{\begin{array}{l} \frac{a^3}{9} + \frac{2aay}{3} + ayy \\ -\frac{a^3}{27} - \frac{aay}{3} - ayy - y^3 \end{array}\right\}$$

$$= \left\{\begin{array}{l} \frac{3a^3}{27} + \frac{2aay}{3} + ayy \\ -\frac{a^3}{27} - \frac{aay}{3} - ayy - y^3 \end{array}\right\} = \frac{2a^3}{27} + \frac{aay}{3} * - y^3.$$

But $ax^2 - x^3$ is $= c$.

Therefore $\frac{2a^3}{27} + \frac{aay}{3} - y^3$ will alſo be $= c$; which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore comes under one, or other, of the three equations $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, and conſequently may be reſolved by means of one of the four Problems above-explained.

Q. E. F.

Art.

Art. CXI. If c is greater than $\frac{2a^3}{27}$, the equation $\frac{2a^3}{27} + \frac{aay}{3} - y^3 = c$ may (by fubtracting $\frac{2a^3}{27}$ from both fides,) be reduced to the equation $\frac{aay}{3} - y^3 = c - \frac{2a^3}{27}$, or $\frac{aay}{3} - y^3 = \frac{27c - 2a^3}{27}$, which may be refolved by Prob. iv; and, if c is lefs than $\frac{2a^3}{27}$, we muft, firft, add y^3 to both fides of the equation $\frac{2a^3}{27} + \frac{aay}{3} - y^3 = c$, by which we fhall obtain the equation $\frac{2a^3}{27} + \frac{aay}{3} = c + y^3$; and then we muft fubtract $\frac{aay}{3}$ from both fides, by which we fhall obtain the equation $\frac{2a^3}{27} = c + y^3 - \frac{aay}{3}$; and, laftly, we muft fubtract c from both fides, by which we fhall obtain the equation $\frac{2a^3}{27} - c = y^3 - \frac{aay}{3}$, or $y^3 - \frac{aay}{3} = \frac{2a^3}{27} - c$, or $y^3 - \frac{aay}{3} = \frac{2a^3 - 27c}{27}$; which is of the fame form as the equation $x^3 - bx = c$, and therefore may be refolved either by Prob. ii. or Prob. iii.

OF

OF CUBICK EQUATIONS THAT HAVE ALL THEIR TERMS COMPLEAT.

Art. CXII. WE muſt now conſider the cubick equations that have all their terms compleat; of which there are the ſeven forms following; to wit,

1ſt, $x^3 + ax^2 + bx = c$,
2dly, $x^3 + ax^2 - bx = c$,
3dly, $x^3 - ax^2 + bx = c$,
4thly, $x^3 - ax^2 - bx = c$,
5thly, $-x^3 + ax^2 + bx = c$,
or $ax^2 + bx - x^3 = c$,
6thly, $-x^3 + ax^2 - bx = c$,
or $ax^2 - x^3 - bx = c$,
and 7thly, $-x^3 - ax^2 + bx = c$,
or $bx - x^3 - ax^2 = c$.

No other combinations of the three quantities x^3, ax^2, and bx with each other, by addition or ſubtraction, are poſſible.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, FROM THE EQUATION $x^3 + ax^2 + bx = c$.

Art. CXIII. THIS equation has evidently but one root; and this one root may be of any magnitude, how great

great or how ſmall ſoever, if the abſolute term c is of a proper magnitude to make it ſo. But, whatever be the magnitude of this root, it will always be poſſible for another quantity to exiſt that ſhall exceed it by a difference equal to $\frac{a}{3}$, or one-third part of a, the co-efficient of xx. Let that quantity be called y. Then we ſhall have $y = x + \frac{a}{3}$, and conſequently $x = y - \frac{a}{3}$. Therefore xx will be $= \overline{y - \frac{a}{3}}^2 \left(= yy - \frac{2ay}{3} + \frac{aa}{9}\right) = yy - \frac{2ay}{3} + \frac{3aa}{27}$, and axx will be $= ayy - \frac{2aay}{3} + \frac{3a^3}{27}$, and x^3 will be $= \overline{y - \frac{a}{3}}^3 \left(= y^3 - 3 \times yy \times \frac{a}{3} + 3 \times y \times \frac{aa}{9} - \frac{a^3}{27}\right)$ $= y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27}$, and bx will be $= b \times \overline{y - \frac{a}{3}}$ $= by - \frac{ab}{3}$; and conſequently $x^3 + ax^2 + bx$ will be $=$

$$\left.\begin{array}{r} y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27} \\ + ayy - \frac{2aay}{3} + \frac{3a^3}{27} \\ + by - \frac{ab}{3} \end{array}\right\} = \begin{array}{l} y^3 - \frac{aay}{3} + \frac{2a^3}{27} \\ + by - \frac{ab}{3}, \end{array}$$

But $x^3 + ax^2 + bx$ is $= c$.

Therefore $\left.\begin{array}{l} y^3 - \frac{aay}{3} + \frac{2a^3}{27} \\ + by - \frac{ab}{3} \end{array}\right\}$ will alſo be $= c$; which

is

is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore will, (when it's terms ſhall have been properly arranged, ſo as to have only the three terms y^3, $\frac{aay}{3}$, and by, which involve the unknown quantity y, remaining on the ſame ſide of the mark of equality,) come under one of the three general forms $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, and conſequently may be reſolved by means of one, or other, of the four Problems above-explained.

Q. E. F.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $x^3 + ax^2 - bx = c$.

Art. CXIV. LET y be a quantity that exceeds the root x of this equation by $\frac{a}{3}$, or the third part of the co-efficient of x^2 in the ſecond term ax^2. Then, ſince y is $= x + \frac{a}{3}$, we ſhall have $x = y - \frac{a}{3}$, and conſequently $xx \left(= \overline{y - \frac{a}{3}}^2\right) = yy - \frac{2ay}{3} + \frac{aa}{9}$, and $ax^2 \left(= ayy - \frac{2aay}{3} + \frac{a^3}{9}\right) = ayy - \frac{2aay}{3} + \frac{3a^3}{27}$, and $x^3 \left(= \overline{y - \frac{a}{3}}^3\right) = y^3 - 3yy \times \frac{a}{3} + 3y \times \frac{aa}{9} - \frac{a^3}{27}$

=

$= y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27}$, and $bx = b \times \sqrt{y - \frac{a}{3}} = by - \frac{ab}{3}$. Therefore $x^3 + ax^2 - bx$ will be $=$

$$\left.\begin{array}{r} y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27} \\ + ayy - \frac{2aay}{3} + \frac{3a^3}{27} \\ - by + \frac{ab}{3} \end{array}\right\} = \left\{\begin{array}{r} y^3 \ * - \frac{2aay}{3} + \frac{2a^3}{27} \\ - by + \frac{ab}{3}. \end{array}\right\}$$

But $x^3 + ax^2 - bx$ is $= c$.

$$\left.\begin{array}{r} \text{Therefore } y^3 - 2aay + \frac{2a^3}{27} \\ - by + \frac{ab}{3} \end{array}\right\} \text{will alſo be } = c\text{; which}$$

is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore may be reſolved by means of one, or other, of the four Problems above-explained.

Q. E. F.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $x^3 - ax^2 + bx = c$.

Art. CXV. WHEN x, or the root of this equation is $= \frac{a}{3}$, we ſhall have $x^2 = \frac{aa}{9}$, and $x^3 = \frac{a^3}{27}$, and ax^2

(=

$\left(=a\times\frac{aa}{9}=\frac{a^3}{9}\right)=\frac{3a^3}{27}$, and $bx=\frac{ab}{3}$ and consequently $x^3-ax^2+bx\left(=\frac{a^3}{27}-\frac{3a^3}{27}+\frac{ab}{3}\right)=\frac{ab}{3}-\frac{2a^3}{27}$. Therefore the abſolute term c, which is equal to x^3-ax^2+bx, will in this caſe be $=\frac{ab}{3}-\frac{2a^3}{27}$. Therefore, while x increaſes from o to $\frac{a}{3}$, the compound quantity x^3-ax^2+bx will have increaſed from o to $\frac{ab}{3}-\frac{2a^3}{27}$, and conſequently will have paſſed through all magnitudes whatſoever that are leſs than $\frac{ab}{3}-\frac{2a^3}{27}$. Therefore, if the abſolute term c is leſs than $\frac{ab}{3}-\frac{2a^3}{27}$, there will be a value of x leſs $\frac{a}{3}$, that will make the compound quantity x^3-ax^2+bx be equal to c; or, in other words, the equation $x^3-ax^2+bx=c$ will have a root that will be leſs than $\frac{a}{3}$. We may therefore in this caſe ſuppoſe x to be equal to $\frac{a}{3}-y$. And, if we make this ſuppoſition, we ſhall have $x^2\left(=\overline{\frac{a}{3}-y}^2=\frac{aa}{9}-\frac{2ay}{3}+yy\right)=\frac{3aa}{27}-\frac{2ay}{3}+yy$, and $x^3\left(=\overline{\frac{a}{3}-y}^3=\frac{a^3}{27}-3\times\frac{aa}{9}\times y+3\times\frac{a}{3}\times yy-y^3\right)=\frac{a^3}{27}-\frac{aay}{3}+ayy-y^3$, and $ax^2=\frac{3a^3}{27}-\frac{2aay}{3}+ayy$, and $bx\left(=\right.$

$b\times$

$\&c \times \frac{a}{3} - y = \frac{ab}{3} - by$, and consequently $x^3 - ax^2 + bx =$

$$\left.\begin{array}{l} \frac{a^3}{27} - \frac{aay}{3} + ayy - y^3 \\ -\frac{3a^3}{27} + \frac{2aay}{3} - ayy \\ +\frac{ab}{3} - by \end{array}\right\} = \left\{\begin{array}{l} \frac{ab}{3} + \frac{aay}{3} * - y^3 \\ -\frac{2a^3}{27} - by. \end{array}\right\}$$

But $x^3 - ax^2 + bx$ is $= c$.

Therefore $\left\{\begin{array}{l} \frac{ab}{3} + \frac{aay}{3} - y^3 \\ -\frac{2a^3}{27} - by \end{array}\right\}$ will alſo be $= c$;

which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore, when it's terms ſhall be properly arranged, (by placing all thoſe which involve the unknown quantity y on one ſide, and all the known terms on the other,) muſt come under one or other of the three forms $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, and conſequently may be reſolved by means of one or other of the four Problems above-explained.

Art. CXVI. Further, when x, after having increaſed from o to $\frac{a}{3}$, increaſes further from being equal to $\frac{a}{3}$ *ad infinitum*, the quantity $x^3 - ax^2 + bx$ will likewiſe increaſe from $\frac{ab}{3} - \frac{2a^3}{27}$ *ad infinitum*, though not without decreaſing during a part of the time of the continual increaſe

increaſe of x, and then increaſing again. Therefore, if the abſolute term c, (to which the compound quantity $x^3 - ax^2 + bx$ is equal,) is of any magnitude greater than $\frac{ab}{3} - \frac{2a^3}{27}$, there will be ſome value of x greater than $\frac{a}{3}$, and, perhaps, more than one ſuch value, that will make the compound quantity $x^3 - ax^2 + bx$ equal to c; or, in other words, the equation $x^3 - ax^2 + bx = c$ will have a root that will be greater than $\frac{a}{3}$. We may therefore in this caſe ſuppoſe x to be equal to $\frac{a}{3} + y$. And, if we make this ſuppoſition, we ſhall have $x^2 \left(= \overline{\frac{a}{3} + y}^2 = \frac{aa}{9} + \frac{2ay}{3} + yy\right) = \frac{3aa}{27} + \frac{2ay}{3} + yy$, and $x^3 \left(= \overline{\frac{a}{3} + y}^3 = \frac{a^3}{27} + 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy + y^3\right) = \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3$, and $ax^2 \left(= a \times x^2 = a \times \overline{\frac{3aa}{27} + \frac{2ay}{3} + yy}\right) = \frac{3a^3}{27} + \frac{2aay}{3} + ayy$, and bx $\left(= b \times \overline{\frac{a}{3} + y}\right) = \frac{ab}{3} + by$, and conſequently

$$x^3 - ax^2 + bx = \left\{\begin{array}{l} \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3 \\ -\frac{3a^3}{27} - \frac{2aay}{3} - ayy \\ +\frac{ab}{3} + by \end{array}\right\} = \left\{\begin{array}{l} \frac{ab}{3} - \frac{aay}{3} + y^3 \\ -\frac{2a^3}{27} + by. \end{array}\right\}$$

But $x^3 - ax^2 + bx$ is $= c$.

Therefore $\left\{\begin{array}{l} \frac{ab}{3} - \frac{aay}{3} + y^3 \\ -\frac{2a^3}{27} + by \end{array}\right\}$ will alſo be $= c$;

which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore, when it's terms ſhall be properly arranged, by placing all thoſe which involve the unknown quantity y on one ſide, and all the known terms on the other, muſt come under one or other of the three forms $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, and conſequently may be reſolved by means of one, or other, of the four Problems above-explained.

It appears therefore that, when c, or the abſolute term of the equation $x^3 - ax^2 + bx$, is leſs than $\frac{ab}{3} - \frac{2a^3}{27}$, one of the values of x in the ſaid equation will be leſs than $\frac{a}{3}$, and may therefore be ſuppoſed to be equal to $\frac{a}{3} - y$, and that by making that ſuppoſition, and ſubſtituting $\frac{a}{3} - y$ inſtead of x in the ſaid equation, it will be converted into the equation $\frac{ab}{3} - \frac{2a^3}{27} + \frac{aay}{3} - by - y^3 = c$, in which the ſquare of the unknown quantity y is wanting, and which therefore may be reſolved by one, or other, of the four Problems above-explained; and it appears likewiſe that, when c, or the abſolute term of the equation $x^3 - ax^2 + bx = c$, is greater than $\frac{ab}{3} - \frac{2a^3}{27}$,

one

one of the values of x in the said equation will be greater than $\frac{a}{3}$, and may therefore be supposed to be equal to $\frac{a}{3} + y$, and that, by making that supposition, and substituting $\frac{a}{3} + y$ instead of x in the equation $x^3 - ax^2 + bx = c$, it will be converted into the equation $\frac{ab}{3} - \frac{2a^3}{27} - \frac{aay}{3} + by + y^3 = c$, in which the square of the unknown quantity y is wanting, and which therefore may be resolved by one, or other, of the four Problems above-explained. Therefore, whatever be the magnitude of the absolute term c, it will always be possible to exterminate the second term ax^2 from the equation $x^3 - ax^2 + bx = c$ by substituting either $\frac{a}{3} - y$ or $\frac{a}{3} + y$ in it's terms instead of x.

Q. E. F.

A SCHOLIUM.

Art. CXVII. This equation $x^3 - ax^2 + bx = c$ is the only form of cubick equations that has sometimes three different roots, or, in the language of the common books of Algebra, three real and affirmative roots. But it has not always three such roots, but sometimes only two such roots, and oftentimes only one. It is, however, always a possible equation, or has at least one such root, of whatever magnitude the absolute term c may happen to be,

and whatever may be the proportions of the co-efficients a and b to each other and to the abſolute term c. But to determine all theſe caſes, and to ſhew when it will have three roots, and what will be the limits of the magnitudes of the ſaid roots, and when it will have two roots, and when it will have only one root, is a matter of a good deal of nicety and difficulty. It has, however, been done with great care and exactneſs for all the different forms of cubick equations that involve the ſquare of the unknown quantity x, as well as for this form $x^3 - ax^2 + bx = c$, (which is the moſt difficult of them, and admits of the greateſt variety of caſes,) in my *Diſſertation on the Uſe of the Negative Sign in Algebra*, from page 64 to page 174. But, as theſe ſubtle inquiries (though very curious) are not abſolutely neceſſary to the extermination of the ſecond term ax^2 from theſe equations, I here omit them.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $x^3 - ax^2 - bx = c$.

Art. CXVIII. IN this equation x^3 is greater than the ſum of both ax^2 and bx, and therefore, *à fortiori*, is greater than ax^2 alone: and conſequently $\frac{x^3}{xx}$ muſt be greater than $\frac{ax^2}{xx}$, that is, x muſt be greater than a, and conſequently,

à for-

à fortiori, it must be greater than $\frac{a}{3}$. Therefore in this equation we may always suppose x to be equal to $\frac{a}{3} + y$. And, if we make this supposition, we shall have $xx \left(= \overline{\frac{a}{3} + y}^2 = \frac{aa}{9} + \frac{2ay}{3} + yy\right) = \frac{3aa}{27} + \frac{2ay}{3} + yy$, and $x^3 \left(= \overline{\frac{a}{3} + y}^3 = \frac{a^3}{27} + 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy + y^3\right) = \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3$, and $ax^2 \left(= a \times x^2 = a \times \overline{\frac{3aa}{27} + \frac{2ay}{3} + yy}\right) = \frac{3a^2}{27} + \frac{2aay}{3} + ayy$, and $bx \left(= b \times \overline{\frac{a}{3} + y}\right) = \frac{ab}{3} + by$,

and consequently $x^3 - ax^2 - bx =$

$$\left.\begin{array}{l} \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3 \\ - \frac{3a^2}{27} - \frac{2aay}{3} - ayy \\ - \frac{ab}{3} - by \end{array}\right\} = \left\{\begin{array}{l} - \frac{2a^3}{27} - \frac{aay}{3} + y^3 \\ - \frac{ab}{3} - by. \end{array}\right\}$$

But $x^3 - ax^2 - bx$ is $= c$.

Therefore $- \frac{2a^3}{27} - \frac{ab}{3} - \frac{aay}{3} - by + y^3$ will also be $= c$, and (adding $\frac{2a^3}{27} + \frac{ab}{3}$ to both sides,) $- \frac{aay}{3} - by + y^3$ will be $= c + \frac{2a^3}{27} + \frac{ab}{3}$, or $y^3 - \overline{\frac{aa}{3} + b} \times y$ will be $= c + \frac{2a^3}{27} + \frac{ab}{3}$, which is an equation in

which the ſquare of the unknown quantity y is wanting, and which comes under the general form $x^3 - bx = c$, and therefore may be reſolved either by Problem ii. and Cardan's ſecond rule, or by Problem iii. and Mr. Raphſon's method of approximation.

Q. E. F.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $-x^3 + ax^2 + bx = c$.

Art. CXIX. TO avoid all the ſubtle inquiries neceſſary to determine the number of roots in this equation $-x^3 + ax^2 + bx = c$, and the limits of their magnitudes, we need only obſerve that ſome one of the values of x in this equation muſt be either equal to $\frac{a}{3}$, or greater than $\frac{a}{3}$, or leſs than $\frac{a}{3}$. If x is equal to $\frac{a}{3}$, we ſhall have $x^2 \left(= \frac{aa}{9}\right) = \frac{3aa}{27}$, and $x^3 = \frac{a^3}{27}$, and $ax^2 = \frac{3a^3}{27}$, and $bx \left(= b \times \frac{a}{3}\right) = \frac{ab}{3}$, and conſequently $-x^3 + ax^2 + bx \left(= -\frac{a^3}{27} + \frac{3a^3}{27} + \frac{ab}{3}\right) = \frac{2a^3}{27} + \frac{ab}{3}$; and therefore, *è converſo*, if $-x^3 + ax^2 + bx$, or, it's equal, the abſolute term c, is equal to $\frac{2a^3}{27} + \frac{ab}{3}$, we may conclude that x is equal to $\frac{a}{3}$.

Art.

Art. CXX. Further, if the abſolute term c is leſs than $\frac{2a^3}{27} + \frac{ab}{3}$, we may conclude that one value of x muſt be leſs than $\frac{a}{3}$. For, ſince, while x increaſes from 0 to $\frac{a}{3}$, the compound quantity $-x^3 + ax^2 + bx$ increaſes from 0 to $\frac{2a^3}{27} + \frac{ab}{3}$, it muſt in it's ſaid increaſe have been ſucceſſively equal to every poſſible quantity that is leſs than $\frac{2a^3}{27} + \frac{ab}{3}$; and therefore, if the abſolute term c, which is equal to $-x^3 + ax^2 + bx$, is leſs than $\frac{2a^3}{27} + \frac{ab}{3}$, there muſt be one value of x leſs than $\frac{a}{3}$, which being ſubſtituted in the ſaid compound quantity $-x^3 + ax^2 + bx$ will make it equal to the abſolute term c, or, in other words, there muſt be one root of the equation $-x^3 + ax^2 + bx = c$ that will be leſs than $\frac{a}{3}$. And therefore in this caſe, or when the abſolute term c is leſs than $\frac{2a^3}{27} + \frac{ab}{3}$, we may ſuppoſe one value of x to be equal to $\frac{a}{3} - y$. And, if we make this ſuppoſition, and ſubſtitute the ſaid compound quantity $\frac{a}{3} - y$ inſtead of x in the equation $-x^3 + ax^2 + bx = c$, we ſhall thereby obtain a new equation involving the letter y inſtead of the letter x, in which the ſquare of the unknown quantity y will be wanting, and which may therefore be reſolved by means of one, or other, of the four Problems above-explained.

plained. For, if x is $= \frac{a}{3} - y$, we ſhall have xx $(= \left[\frac{a}{3} - y\right]^2 = \frac{aa}{9} - \frac{2ay}{3} + yy) = \frac{3aa}{27} - \frac{2ay}{3} + yy$, and axx $(= a \times \frac{3aa}{27} - \frac{2ay}{3} + yy) = \frac{3a^3}{27} - \frac{2aay}{3} + ayy$, and x^3 $(= \left[\frac{a}{3} - y\right]^3 = \frac{a^3}{27} - 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy - y^3) = \frac{a^3}{27} - \frac{aay}{3} + ayy - y^3$, and bx $(= b \times \frac{a}{3} - y) = \frac{ab}{3} - by$, and conſequently $- x^3 + ax^2 + bx =$

$$\left\{\begin{array}{l} -\frac{a^3}{27} + \frac{aay}{3} - ayy + y^3 \\ +\frac{3a^3}{27} - \frac{2aay}{3} + ayy \\ +\frac{ab}{3} - by \end{array}\right\} = \left\{\begin{array}{l} +\frac{2a^3}{27} - \frac{aay}{3} + y^3 \\ +\frac{ab}{3} - by. \end{array}\right\}$$

But $- x^3 + ax^2 + bx$ is $= c$.

Therefore $\frac{2a^3}{27} + \frac{ab}{3} - \frac{aay}{3} - by + y^3$ will alſo be $= c$; which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore may be reſolved by means of one, or other, of the four Problems above-explained.

Q. E. F.

Art.

Art. CXXI. And, if the abſolute term c is greater than $\frac{2a^3}{27} + \frac{ab}{3}$, we are ſure that x muſt either be leſs than $\frac{a}{3}$, or greater than $\frac{a}{3}$. Let us therefore, firſt, ſuppoſe it to be equal to $\frac{a}{3} - y$, and ſubſtitute $\frac{a}{3} - y$ in it's ſtead in the propoſed equation $-x^3 + ax^2 + bx = c$; and we ſhall thereby convert the ſaid equation into the equation $\frac{2a^3}{27} + \frac{ab}{3} - \frac{aay}{3} - by + y^3 = c$, (as we have already ſeen,) which (becauſe c is ſuppoſed to be greater than $\frac{2a^3}{27} + \frac{ab}{3}$,) may be reduced to the equation $y^3 - \frac{aay}{3} - by = c - \frac{2a^3}{27} - \frac{ab}{3}$. Now, if this equation is a poſſible equation, the ſuppoſition from which it was derived, to wit, that x was $= \frac{a}{3} - y$, or that x was leſs than $\frac{a}{3}$, muſt alſo be poſſible. But the equation $y^3 - \frac{aay}{3} - by = c - \frac{2a^3}{27} - \frac{ab}{3}$ is of the ſame form with the equation $x^3 - bx = c$; which is always poſſible, whatever be the magnitudes of b and c; and therefore the ſaid equation $y^3 - \frac{aay}{3} - by = c - \frac{2a^3}{27} - \frac{ab}{3}$ is poſſible. Therefore, whether the abſolute term c of the propoſed equation $-x^3 + ax^2 + bx = c$ be leſs, or be greater, than $\frac{2a^3}{27} + \frac{ab}{3}$, we may in either caſe exterminate it's ſecond term ax^2, or reduce it to an equation in which the

ſquare

ſquare of the unknown quantity ſhall be wanting, by ſuppoſing x to be equal to $\frac{a}{3} - y$, and ſubſtituting the ſaid compound quantity $\frac{a}{3} - y$ inſtead of x in the terms of the propoſed equation $-x^3 + ax^2 + bx = c$.

Q. E. F.

But, 2ndly, if we ſuppoſe x to be greater than $\frac{a}{3}$, and ſubſtitute $\frac{a}{3} + y$ inſtead of it in the propoſed equation $-x^3 + ax^2 + bx = c$, that ſubſtitution will likewiſe anſwer the purpoſe of exterminating it's ſecond term ax^2, or converting it into another equation in which the ſquare of the unknown quantity ſhall be wanting. For, if we ſuppoſe x to be $= \frac{a}{3} + y$, we ſhall have $x^2 (= \overline{\frac{a}{3} + y}^2 = \frac{aa}{9} + \frac{2ay}{3} + yy) = \frac{3aa}{27} + \frac{2ay}{3} + yy$, and $ax^2 = \frac{3a^3}{27} + \frac{2aay}{3} + ayy$, and $x^3 (= \overline{\frac{a}{3} + y}^3 = \frac{a^3}{27} + 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy + y^3) = \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3$, and $bx (= b \times \overline{\frac{a}{3} + y}) = \frac{ab}{3} + by$, and conſequently $-x^3 + ax^2 + bx =$

$$\left\{\begin{array}{l} -\frac{a^3}{27} - \frac{aay}{3} - ayy - y^3 \\ +\frac{3a^3}{27} + \frac{2aay}{3} + ayy \\ +\frac{ab}{3} + by \end{array}\right\} = \left\{\begin{array}{l} \frac{2a^3}{27} + \frac{aay}{3} * - y^3 \\ +\frac{ab}{3} + by. \end{array}\right.$$

But

But $-x^3 + ax^2 + bx$ is $= c$.

Therefore $\frac{2a^3}{27} + \frac{ab}{3} + \frac{aay}{3} + by * - y^3$ will alſo be $= c$; and (ſubtracting $\frac{2a^3}{27} + \frac{ab}{3}$ from both ſides,) $\frac{aay}{3} + by - y^3$ will be $= c - \frac{2a^3}{27} - \frac{ab}{3}$; which is an equation in which the ſquare of the unknown quantity y is wanting, and which therefore, if it be poſſible, (for equations of this form, or of the general form $bx - x^3 = c$, are not always poſſible, but only when the abſolute term c is not greater than $\frac{2b\sqrt{b}}{3\sqrt{3}}$, or $\frac{cc}{4}$ is not greater than $\frac{b^3}{27}$,) may be reſolved by means of one, or other, of the four Problems above-explained.

Q. E. F.

Art. CXXII. Therefore, whether the abſolute term c be greater than, or leſs than $\frac{2a^3}{27} + \frac{ab}{3}$, it will always be poſſible, by ſuppoſing one of the values of x in the propoſed equation $-x^3 + ax^2 + bx = c$ to be equal either to $\frac{a}{3} - y$ or to $\frac{a}{3} + y$, and ſubſtituting one of thoſe compound quantities inſtead of x in the terms of the ſaid equation, to exterminate the ſecond term ax^2 of the ſaid equation, or to convert it into another equation in which the ſquare of the unknown quantity y ſhall be wanting, and which therefore may be reſolved by one, or other, of the four Problems above-explained.

OF

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $-x^3 + ax^2 - bx = c$.

Art. CXXIII. IN this equation, if we ſubſtitute either $\frac{a}{3} + y$, or $\frac{a}{3} - y$ inſtead of x, the equation thereby obtained will be an equation in which the ſquare of the unknown quantity y will be wanting. For, if x is $= \frac{a}{3} + y$, we ſhall have $xx \left(= \overline{\frac{a}{3} + y}^2\right) = \frac{aa}{9} + \frac{2ay}{3} + yy) = \frac{3aa}{27} + \frac{2ay}{3} + yy$, and $axx = \frac{3a^3}{27} + \frac{2aay}{3} + ayy$, and $x^3 \left(= \overline{\frac{a}{3} + y}^3\right) = \frac{a^3}{27} + 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy + y^3) = \frac{a^3}{27} + \frac{aay}{3} + ayy + y^3$, and $bx \left(= b \times \overline{\frac{a}{3} + y}\right) = \frac{ab}{3} + by$; and conſequently $-x^3 + axx - bx$ will be $=$

$$\left\{\begin{array}{l} -\frac{a^3}{27} - \frac{aay}{3} - ayy - y^3 \\ +\frac{3a^3}{27} + \frac{2aay}{3} + ayy \\ -\frac{ab}{3} - by \end{array}\right\} = \left\{\begin{array}{l} \frac{2a^3}{27} + \frac{aay}{3} - y^3 \\ -\frac{ab}{3} - by. \end{array}\right\}$$

But

But $-x^3 + ax^2 - bx$ is $= c$.

Therefore $\frac{2a^3}{27} - \frac{ab}{3} + \frac{aay}{3} - by - y^3$ will alſo be $= c$; which is an equation in which the ſquare of the unknown quantity y is wanting. And, if x is $= \frac{a}{3} - y$, we ſhall have $xx \left(= \overline{\frac{a}{3} - y}^2 = \frac{aa}{9} - \frac{2ay}{3} + yy\right) = \frac{3aa}{27} - \frac{2ay}{3} + yy$, and $axx = \frac{3a^3}{27} - \frac{2aay}{3} + ayy$, and $x^3 \left(= \overline{\frac{a}{3} - y}^3 = \frac{a^3}{27} - 3 \times \frac{aa}{9} \times y + 3 \times \frac{a}{3} \times yy - y^3\right) = \frac{a^3}{27} - \frac{aay}{3} + ayy - y^3$, and $bx \left(= b \times \overline{\frac{a}{3} - y}\right) = \frac{ab}{3} - by$; and conſequently $-x^3 + ax^2 - bx$ will be $=$

$$\left\{\begin{array}{l} -\frac{a^3}{27} + \frac{aay}{3} - ayy + y^3 \\ +\frac{3a^3}{27} - \frac{2aay}{3} + ayy \\ -\frac{ab}{3} + by \end{array}\right\} = \left\{\begin{array}{l} \frac{2a^3}{27} - \frac{aay}{3} + y^3 \\ -\frac{ab}{3} + by. \end{array}\right\}$$

But $-x^3 + ax^2 - bx$ is $= c$.

Therefore $\frac{2a^3}{27} - \frac{ab}{3} - \frac{aay}{3} + by + y^3$ will alſo be $= c$; which is an equation in which the ſquare of the unknown quantity y is wanting.

Now

Now the value of x in the propoſed equation $-x^3 + ax^2 - bx = c$, or, if it has more than one value, each of it's values, muſt be either greater than $\frac{a}{3}$ or leſs than $\frac{a}{3}$, and therefore may be ſuppoſed to be equal either to $\frac{a}{3} + y$ or to $\frac{a}{3} - y$; and conſequently either the equation $\frac{2a^3}{27} - \frac{ab}{3} + \frac{aay}{3} - by - y^3 = c$ (ariſing from the ſuppoſition that x is equal to $\frac{a}{3} + y$,) or the equation $\frac{2a^3}{27} - \frac{ab}{3} - \frac{aay}{3} + by + y^3 = c$, (ariſing from the ſuppoſition that x is equal to $\frac{a}{3} - y$) muſt be a poſſible equation. And it may happen that both theſe equations may be poſſible, namely, if the propoſed equation $-x^3 + ax^2 - bx = c$ has two roots, of which the leſſer is leſs than $\frac{a}{3}$, and the the greater is greater than $\frac{a}{3}$. We muſt therefore firſt examine the equation $\frac{2a^3}{27} - \frac{ab}{3} + \frac{aay}{3} - by - y^3 = c$, (which aroſe from the ſuppoſition that x was equal to $\frac{a}{3} + y$) to find, whether, or not, it is poſſible; and, if we find it is poſſible, our buſineſs is done, or we have converted the propoſed equation $-x^3 + ax^2 - bx = c$ into another equation in which the ſquare of the unknown quantity is wanting, and which therefore may be reſolved by means of one, or other, of the four

Problems

Problems above-explained. And, if we find this firſt equation to be impoſſible, the other equation $\frac{2a^3}{27} - \frac{ab}{3} - \frac{aay}{3} + by + y^3 = c$ (which aroſe from the ſuppoſition that x was equal to $\frac{a}{3} - y$,) will be poſſible, and will anſwer the object we were in purſuit of, namely, that of exterminating the ſecond term ax^2 from the propoſed equation $-x^3 + ax^2 - bx = c$, or converting the ſaid equation into another equation in which the ſquare of the unknown quantity is wanting, and which therefore may be reſolved by means of one, or other, of the four Problems above-explained.

Q. E. F.

Art. CXXIV. If we find both the equations $\frac{2a^3}{27} - \frac{ab}{3} + \frac{aay}{3} - by - y^3 = c$ and $\frac{2a^3}{27} - \frac{ab}{3} - \frac{aay}{3} + by + y^3 = c$ to be poſſible, we muſt conſider which of the two may be reſolved with the leaſt trouble, and muſt reſolve that alone (without taking any further notice of the other equation,) by the method preſcribed for that purpoſe in one, or other, of the four Problems above-explained; and when we have, by this means, obtained one value of x, or one root of the propoſed equation $-x^3 + ax^2 - bx = c$, we ſhall be able, by the help of this one root, to reduce the ſaid equation to a quadratick equation, which will involve it's other root; and then, by the reſolution of this quadratick equation (which is always much eaſier than the reſolution of any cubick equation,) we

we may find the ſaid remaining root of the propoſed equation $-x^3 + ax^2 - bx = c$.

Q. E. I.

OF THE EXTERMINATION OF THE SECOND TERM, ax^2, OF THE EQUATION $-x^3 - ax^2 + bx = c$.

Art. CXXV. TO exterminate the ſecond term of this equation $-x^3 - ax^2 + bx = c$ we muſt ſuppoſe x to be equal to $y - \frac{a}{3}$; which is a ſuppoſition that is evidently poſſible, becauſe whatever be the magnitude of x, there will always be ſome other quantity, which we may call y, which will exceed it by the given difference $\frac{a}{3}$. Now, if we ſuppoſe x to be $= y - \frac{a}{3}$, we ſhall have xx $(= \overline{y - \frac{a}{3}}^2 = yy - \frac{2ay}{3} + \frac{aa}{9}) = yy - \frac{2ay}{3} + \frac{3aa}{27}$, and $axx = ayy - \frac{2aay}{3} + \frac{3a^3}{27}$, and x^3 $(= \overline{y - \frac{a}{3}}^3 = y^3 - 3 \times yy \times \frac{a}{3} + 3 \times y \times \frac{aa}{9} - \frac{a^3}{27} = y^3 - ayy + \frac{aay}{3} - \frac{a^3}{27}$, and bx $(= b \times \overline{y - \frac{a}{3}}) = by - \frac{ab}{3}$; and

and consequently $-x^3 - ax^2 + bx$ will be =

$$\left\{\begin{array}{r} -y^3 + ayy - \frac{aay}{3} + \frac{a^3}{27} \\ -ayy + \frac{2aay}{3} - \frac{3a^3}{27} \\ +by - \frac{ab}{3} \end{array}\right\} = \left\{\begin{array}{r} -y^3 + \frac{aay}{3} - \frac{2a^3}{27} \\ +by - \frac{ab}{3} \end{array}\right\}$$

But $-x^3 - ax^2 + bx$ is $= c$.

Therefore $-y^3 + \frac{aay}{3} + by - \frac{2a^3}{27} - \frac{ab}{3}$ will alſo be $= c$, and conſequently (adding $\frac{2a^3}{27} + \frac{ab}{3}$ to both ſides,) $-y^3 + \frac{aay}{3} + by$ will be $= c + \frac{2a^3}{27} + \frac{ab}{3}$, or $\frac{aay}{3} + by - y^3$ will be $= c + \frac{2a^3}{27} + \frac{ab}{3}$, or $\frac{aa}{3} + b \times y - y^3$ will be $= c + \frac{2a^3}{27} + \frac{ab}{3}$; which is an equation in which the ſquare of the unknown quantity y is wanting, and which is of the ſame form as the equation $bx - x^3 = c$, and therefore may be reſolved by means of Problem iv.

Q. E. F.

Art. CXXVI. This equation $-x^3 - ax^2 + bx = c$ is the laſt of the ſeven forms of cubick equations that have all their terms compleat, which were ſet down above in Art. 112; and therefore we have now ſhewn, how, by ſuppoſing x in ſome caſes to be equal to the compound

quantity $y - \frac{a}{3}$, and in others to $\frac{a}{3} - y$, and in others to $\frac{a}{3} + y$, we may always convert a cubick equation that involves in it the ſquare of the unknown quantity x, into another cubick equation, (involving the unknown quantity y inſtead of the firſt unknown quantity x,) in-which the ſquare of the ſaid unknown quantity y ſhall be wanting, and which therefore may be reſolved by means of one, or other, of the four Problems above-explained. And this is all that is abſolutely neceſſary to be known in order to enable the reader to find the roots of any propoſed cubick equation whatſoever; which is all that I propoſed to ſhew in this Appendix. But the more curious and inquiſitive ſtudents of this ſcience of Algebra, (for, upon the principles of Mr. Frend's book and this Appendix, from which all the nonſenſe of negative quantities, or quantities leſs than nothing, or quantities ariſing from the ſubtraction of a greater quantity from a leſſer, is compleatly excluded, we may venture with propriety to call Algebra *a Science*, and one by which, in the words that have been cited from Des Cartes in the motto to this Appendix, *Ingenium excolatur et perſpicacius reddatur*,) will probably be deſirous of knowing more exactly the limits of the magnitudes of the abſolute terms of ſome of the forms of cubick equations, (as for example, the cubick equations $-x^3 + ax^2 + bx = c$, and $-x^3 - ax^2 + bx = c$,) within which only the ſaid equations are poſſible, and the number of roots which ſome of theſe equations will admit of, and the limits of the magnitudes of each of the ſaid roots; all which particulars have, in this Appendix, been fully ſet forth only with

reſpect to the three equations $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, which are the ſubjects of the four Problems above-explained. And therefore, for the further ſatisfaction of ſuch curious readers with reſpect to theſe particulars in all the other forms of cubick equations, I muſt refer them to my Diſſertation on the Uſe of the Negative Sign in Algebra, pages 64, 65, 66, &c - - - to page 174, where the matter is very fully and diſtinctly conſidered and explained in all it's various branches and ſubdiviſions.

OF THE RESOLUTION OF CUBICK EQUATIONS THAT HAVE ALL THEIR TERMS COMPLEAT, BY MR. RAPHSON'S METHOD OF APPROXIMATION, WITHOUT FIRST EXTERMINATING THE SQUARE OF THE UNKNOWN QUANTITY.

Art. CXXVII. IN the foregoing articles it has been ſhewn that all ſorts of cubick equations that involve in them the ſquare of the unknown quantity x, may be converted into other cubick equations in which the ſquare of the unknown quantity, that is the root of ſuch ſecond equation, is wanting. This converſion, it has been ſhewn, is performed by ſuppoſing x to be equal either to the compound quantity $y - \frac{a}{3}$, or to the compound quan-

tity $\frac{a}{3} - y$, or to the compound quantity $\frac{a}{3} + y$, as the caſe may require, and then ſubſtituting the ſaid compound quantity inſtead of x in the original equation. For the new equation ariſing from this ſubſtitution, and of which y will be the root, will always (as we have ſeen,) be without the ſquare of the new unknown quantity y. This new equation therefore will always come under one of theſe three forms, $x^3 + bx = c$, $x^3 - bx = c$, and $bx - x^3 = c$, or $y^3 + qy = r$, $y^3 - qy = r$, and $qy - y^3 = r$, and therefore may be reſolved by means of one, or other, of the four Problems above-explained. And, when the value of y is diſcovered by that means, the value of x in the original equation may be deduced from it by ſubtracting $\frac{a}{3}$ from y, if x was equal to $y - \frac{a}{3}$, or by ſubtracting y from $\frac{a}{3}$, if x was equal to $\frac{a}{3} - y$, or, laſtly, by adding y to $\frac{a}{3}$, if x was equal to $\frac{a}{3} + y$. And, if the ſecond equation (which involves the unknown quantity y,) comes under the general equation $y^3 + qy = r$, the value of y may be obtained to the intended degree of exactneſs at once, or by a ſingle proceſs, or ſet of arithmetical operations, by means of one of the three expreſſions given by Cardan's firſt rule, which was invented by Scipio Ferreus, and which is explained above in the ſolution of Problem i: and, if the ſaid ſecond equation comes under the firſt caſe of the general equation $y^3 - qy = r$, or that in which the abſolute term r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$,

or

or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, the value of y may be obtained to the intended degree of exactneſs at once, or by a ſingle proceſs, or ſet of arithmetical operations, by means of one of the three expreſſions given by Cardan's ſecond rule, which was invented by Nicholas Tartalea, and which is explained above in the ſolution of Problem ii: but, if the ſaid ſecond equation comes under the ſecond caſe of the general equation $y^3 - qy = r$, in which r is leſs than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, or if it comes under the general equation $qy - y^3 = r$, we ſhall not be able to obtain the value of y to the intended degree of exactneſs at once, or by a ſingle proceſs, or ſet of arithmetical operations, by means of either of Cardan's rules, but ſhall be obliged to employ repeated proceſſes of Mr. Raphſon's method of approximation for that purpoſe, in the manner deſcribed in Problems iii and iv. And in theſe latter caſes it will, I believe, be generally found moſt convenient to reſort at once to Mr. Raphſon's method of approximation for the reſolution of the firſt equation, (of which x is the root,) inſtead of firſt reducing it to one of the two forms $y^3 - qy = r$ and $qy - y^3 = r$, and then reſolving the ſaid ſecond equation by Mr. Raphſon's method: and this, even though the firſt equation ſhould have all it's terms compleat, and ſhould alſo be of the form $x^3 - ax^2 + bx = c$, which is the moſt complicated and difficult of all the forms of cubick equations, and ſometimes admits of three real and poſitive roots. And even in the former caſes, which may be made to come ultimately under one, or other, of Cardan's rules, I believe it will often be found

to be moſt convenient to reſort at once to Mr. Raphſon's method of approximation for the reſolution of the firſt equation. Of ſuch a reſolution I ſhall therefore now proceed to give one or two examples.

AN EXAMPLE OF THE RESOLUTION OF A CUBICK EQUATION THAT HAS ALL IT'S TERMS COMPLEAT, BY MR. RAPHSON'S METHOD OF APPROXIMATION, WITHOUT FIRST EXTERMINATING THE SQUARE OF THE UNKNOWN QUANTITY.

Art. CXXVIII. LET it be propoſed to reſolve by Mr. Raphſon's method of approximation the cubick equation $x^3 - 3xx - 36x = 342$.

Here we may obſerve, in the firſt place, that, if x is equal to 10, xx will be $= 100$, and x^3 will be $= 1000$, and $3xx$ will be $= 300$, and $36x$ will be $(= 36 \times 10) = 360$, and conſequently that $x^3 - 3xx - 36x$ will be $(= 1000 - 300 - 360 = 1000 - 660) = 340$; which is but a little leſs than 342, or the abſolute term of the propoſed equation $x^3 - 3xx - 36x = 342$. Therefore 10 will be a little leſs than the true value of x in that equation.

Let

Let us therefore, in the ſecond place, ſuppoſe x to be $= 10 + z$.

Then we ſhall have xx $(= \overline{10 + z}^2 = 100 + 2 \times 10 \times z + \&c) = 100 + 20z + \&c$, and x^3 $(= \overline{10 + z}^3 = 1000 + 3 \times 100 \times z + \&c) = 1000 + 300z + \&c$, and $3xx$ $(= 3 \times \overline{100 + 20z + \&c} = 300 + 60z + \&c$, and $36x$ $(= 36 \times \overline{10 + z} = 36 \times 10 + 36 \times z) = 360 + 36z$.

Therefore $x^3 - 3xx - 36x$ will be $=$

$$\left\{\begin{array}{rrl} 1000 & + 300 \times z & + \&c \\ - 300 & - 60 \times z & - \&c \\ - 360 & - 36 \times z & \end{array}\right\} = 340 + 204 \times z \ \&c.$$

But $x^3 - 3xx - 36x$ is $= 342$.

Therefore $340 + 204 \times z$ &c will alſo be $= 342$. And conſequently (ſubtracting 340 from both ſides,) $204 \times z$ will be $= 2$, and z will be $= \frac{2}{204} = \frac{1}{102}\Big) = 0.009,803$, &c. Therefore x, or $10 + z$, will be $(= 10 + 0.009, 803, \&c) = 10.009,803, \&c$; that is, the root of the propoſed equation $x^3 - 3xx - 36x = 342$ will be nearly equal to 10.009,803, &c.

Q. E. I.

Art. CXXIX. This value of x is pretty near the truth, as will appear upon trial. For, if we ſuppoſe x to be $= 10.009,803$, we ſhall have $xx = 100.196,156,98$,

809, and $x^3 = 1002.943,783,906,326,624,627$, and $3xx = 300.588,468,296,427$, and $36x$ ($= 36 \times 10.009,803$) $= 360.352,908$, and $x^3 - 3xx - 36x =$

$$\left\{\begin{array}{l} 1002.943,783,906,326,624,627 \\ -\ 300.588,468,296,427 \\ -\ 360.352,908 \end{array}\right\} =$$

$$\left\{\begin{array}{l} 1002.943,783,906,326,624,627 \\ -\ 660.941,376,296,427, \end{array}\right\} = 342.002,407,$$

609,899,624,627; which is greater than 342 by less than 0.00241, or the 141,742d part of 342.

Art. CXXX. I believe this manner of obtaining the value of x in the proposed equation $x^3 - 3xx - 36x = 342$ to be much less troublesome than, first, to convert this equation into an equation of one of the three forms $y^3 + qy = r$, $y^3 - qy = r$, and $qy - y^3 = r$, by supposing x to be equal to $\left(\frac{a}{3} + y\right.$, or $\frac{3}{3} + y$, or$)$ $1 + y$, and substituting $1 + y$ instead of it in the proposed equation $x^3 - 3xx - 36x = 342$, whereby the said equation would be changed into the equation $-38 - 39y * + y^3 = 342$, or $y^3 - 39y = 380$, (in which the square of the unknown quantity y is wanting,) and then resolving the said second equation $y^3 - 39y = 380$, thereby obtained, by means of one of the expressions of Cardan's second rule, under which the said equation falls. But I leave the trial and the comparison of these two methods of resolving the proposed equation $x^3 - 3xx - 36x = 342$ to the industry of the reader.

ANOTHER

ANOTHER EXAMPLE OF THE RESOLUTION OF A CUBICK EQUATION THAT HAS ALL IT'S TERMS COMPLEAT, BY MR. RAPHSON'S METHOD OF APPROXIMATION, WITHOUT FIRST EXTERMINATING THE SQUARE OF THE UNKNOWN QUANTITY.

Art. CXXXI. LET the equation that is to be reſolved by this method of approximation be $x^3 + 15.75x^2 + 43.6875x = 440.046,875$.

Now it is evident, in the firſt place, that this equation can have but one root; and this root may be found by a few eaſy conjectures and trials to be nearly equal to, but ſomewhat leſs than, 3.79.

For, if x is ſuppoſed to be $= 1$, we ſhall have $xx = 1$, and x^3 alſo $= 1$, and conſequently $x^3 + 15.75x^2 + 43.6875x$ $(= 1 + 15.75 \times 1 + 43.6875 \times 1 = 1 + 15.75 + 43.6875 = 16.75 + 43.6875) = 60.4375$; which is much leſs than the abſolute term 440.046,875. Therefore 1 is much leſs than the true value of x in that equation. We may therefore reaſonably conjecture that the ſaid true value will not be leſs than 3, and, for a ſecond trial, will ſuppoſe it to be equal to 3.

Now, if x be $= 3$, we ſhall have $xx = 9$, and $x^3 = 27$, and conſequently $x^3 + 15.75 \times xx + 43.6875 \times x$ will be

be (= 27 + 15.75 × 9 + 43.6875 × 3 = 27 + 141.75 + 131.0625) = 299.8125; which is ſtill conſiderably leſs than 440.046,875, or the abſolute term of the equation $x^3 + 15.75xx + 43.6875x = 440.046,875$. Therefore 3 is leſs than the true value of x in that equation.

We will therefore in the next place ſuppoſe x to be = 4.

Then will xx be = 16, and x^3 be = 64, and $x^3 + 15.75 \times xx + 43.6875 \times x$ will be (= 64 + 15.75 × 16 + 43.6875 × 4 = 64 + 252.00 + 174.7500) = 490.7500; which is greater than 440.046,875, or the abſolute term of the equation $x^3 + 15.75xx + 43.6875x = 440.046,875$. Therefore the true value of x in this equation muſt be leſs than 4.

We therefore now know that the true value of x is greater than 3, but leſs than 4. We muſt therefore, in the next place, ſuppoſe it to be equal to ſome intermediate number between 3 and 4.

Now, ſince, when x is equal to 3, the compound quantity $x^3 + 15.75xx + 43.6875x$ is equal to 299.8125, or, nearly, to 300; and when x is equal to 4, the ſame compound quantity is equal to 490.7500, or, nearly to 490; and, when x is equal to the root of the propoſed equation, the ſame compound quantity is equal to 440.046,875, or, nearly, to 440; we may reaſonably conjecture that the exceſs of 4 above 3 will be to the exceſs of 4 above the root of the propoſed equation in nearly the ſame proportion

tion as the exceſs of 490 above 300 is to the exceſs of 490 above 440, that is, that 1 will be to the exceſs of 4 above the root of the ſaid equation nearly as 190 is to 50, or as 19 is to 5; whence it will follow that the exceſs of 4 above the root of the ſaid equation will be nearly equal to $\left(\frac{5 \times 1}{19}\right.$, or to $\frac{5}{19}$, or) to 0.21, and conſequently that the ſaid root will be nearly equal to $4 - 0.21$, or 3.79. We will therefore ſuppoſe x to be equal to 3.79, and try what will be the value of the compound quantity $x^3 + 15.75xx + 43.6875x$ reſulting from this ſuppoſition.

Now, if x is $= 3.79$, we ſhall have xx $(= \overline{3.79}^2) = 14.3641$, and x^3 $(= \overline{3.79}^3) = 54.439,939$, and $15.75xx$ $(= 15.75 \times 14.3641) = 226.234,575$, and $43.6875x$ $(= 43.6875 \times 3.79) = 165.575,625$, and conſequently $x^3 + 15.75xx + 43.6875x$ $(= 54.439,939 + 226.234,575 + 165.575,625) = 446.250,139$; which is a little greater than 440.046,875, or the abſolute term of the propoſed equation $x^3 + 15.75xx + 43.6875x = 440.046,875$. Therefore 3.79 will be a little greater than the true root of that equation: but it will be near enough to the ſaid root to be an excellent baſis of a further approximation to it in the method taught by Mr. Raphſon.

Art. CXXXII. Let us therefore now ſuppoſe x, or the root of the equation $x^3 + 15.75xx + 43.6875x = 440.046,875$, to be equal to $3.79 - z$, and let this compound quantity $3.79 - z$ be ſubſtituted inſtead of x in the terms of the ſaid equation.

Then

Then will xx be $(= \overline{3.79 - z}^2 =$

$\overline{3.79}^2 - 2 \times 3.79 \times z + \&c = \overline{3.79}^2 - 7.58 \times z + \&c)$

$= 14.3641 - 7.58 \times z + \&c,$

and x^3 will be $(= \overline{3.79 - z}^3 = \overline{3.79}^3$

$- 3 \times \overline{3.79}^2 \times z + \&c = \overline{3.79}^3 - 3 \times$

$14.3641 \times z + \&c = \overline{3.79}^3 - 43.0923 \times z + \&c)$

$= 54.439,939 - 43.0923 \times z + \&c,$

and $15.75xx$ will be $(= 15.75 \times \overline{14.3641 - 7.58 \times z} + \&c$

$= 15.75 \times 14.3641 - 15.75 \times 7.58 \times z + \&c,)$

$= 226.234,575 - 119.3850 \times z + \&c,$

and $43.6875x$ will be $(= 43.6875 \times \overline{3.79 - z}$

$= 43.6875 \times 3.79 - 43.6875 \times z)$

$= 165.575,625 - 43.6875 \times z;$

and consequently the trinomial quantity $x^3 + 15.75xx + 43.6875x$ will be $=$

$$\left.\begin{array}{rrl} & 54.439,939 - & 43.0923 \times z + \&c \\ + & 226.234,575 - & 119.3850 \times z + \&c \\ + & 165.575,625 - & 43.6875 \times z \end{array}\right\} =$$

$$446.250,139 - 206.1648 \times z + \&c.$$

But the trinomial quantity $x^3 - 15.75xx + 43.6875x$ is $= 440.046,875.$

Therefore $446.250,139 - 206.1648 \times z + \&c$ will alſo be $= 440.046,875.$ And conſequently (adding $206.1648 \times z$ to both ſides,) $446.250,139$ will be $= 440.046,$

875

875 + 206.1648 × z, and (ſubtracting 440.046,875 from both ſides,) 206.1648 × z will be = 6.203,264, and conſequently z will be $(= \frac{6.203,264}{206.1648}) = 0.030,08$. Therefore x, or $3.79 - z$, will be $= 3.79 - 0.030,08 = 3.759,92$, or, very nearly, 3.760; that is, the root of the propoſed equation $x^3 + 15.75xx + 43.6875x = 440.046,875$ will be, very nearly, = 3.760.

Q. E. I.

Art. CXXXIII. Now that this number, 3.760, is very nearly equal to the true value of x in the equation $x^3 + 15.75xx + 43.6875x = 440.046,875$ will appear upon trial, as follows:

If x is = 3.760, we ſhall have xx $(= \overline{3.760}^2) = 14.137,600$, and x^3 $(= \overline{3.760}^3) = 53.157,376,000$, and $15.75xx$ $(= 15.75 \times 14.137,600) = 222.667,200,00$, and $43.6875x$ $(= 43.6875 \times 3.760) = 164.265,000,0$, and conſequently $x^3 + 15.75xx + 43.6875x$ $(= 53.157,376,000 + 222.667,200,00 + 164.265,000,0 = 440.089,576$; which exceeds 440.046,875, or the abſolute term of the equation $x^3 + 15.75xx + 43.6875x = 440.046,875$, by only the ſmall quantity 0.042,701, which is leſs than the 10,305th part of the ſaid abſolute term 440.046,875. This is a very great degree of exactneſs for the reſult of only one proceſs of Mr. Raphſon's method of approximation. A ſecond proceſs of this method, by ſuppoſing x to be equal to $3.760 - z$, and ſubſtituting this compound quantity inſtead of x in the propoſed equation $x^3 + 15.75xx + 43.6875x = 440.046,875$, would give us a value of x that would

would be exact in the firſt eight figures. But this I ſhall leave to the induſtry of the reader, as well as the buſineſs of reſolving this equation in the other way, by firſt converting it into another cubick equation in which the ſquare of the unknown quantity ſhall be wanting, and then reſolving the ſaid ſecond equation by means of one of the four Problems above-explained; which method of reſolving it I believe he will find much more laborious than the foregoing reſolution of it by three, or four, conjectures and trials of very eaſy numbers conſiſting only of one figure, and, afterwards, by only one proceſs of Mr. Raphſon's method of approximation. But of this the reader will be beſt able to judge when he has reſolved this and a few more ſuch equations by both theſe methods.

I therefore think it unneceſſary to add any thing further concerning the reſolution of cubick equations.

OF THE RESOLUTION OF BIQUADRATICK EQUATIONS, OR EQUATIONS OF THE FOURTH ORDER, OR THAT INVOLVE THE FOURTH POWER OF THE UNKNOWN QUANTITY.

Art. CXXXIV. THE number of all the different forms of biquadratick equations, or equations that involve the fourth power of the unknown quantity, is no leſs than 46, the ſaid forms being as follows:

1.

1. $x^4 = s.$

2. $x^4 + rx = s.$

3. $x^4 - rx = s.$

4. $-x^4 + rx = s,$

or $rx - x^4 = s.$

5. $x^4 + qx^2 = s.$

6. $x^4 - qx^2 = s.$

7. $-x^4 + qx^2 = s,$

or $qx^2 - x^4 = s.$

8. $x^4 + px^3 = s.$

9. $x^4 - px^3 = s.$

10. $-x^4 + px^3 = s,$

or $px^3 - x^4 = s.$

11. $x^4 + qx^2 + rx = s.$

12. $x^4 + qx^2 - rx = s.$

13. $x^4 - qx^2 + rx = s.$

14. $x^4 - qx^2 - rx = s.$

15. $-x^4 + qx^2 + rx = s.$

16. $-x^4 + qx^2 - rx = s.$

17. $-x^4 - qx^2 + rx = s.$

18. $x^4 + px^3 + rx = s.$

19. $x^4 + px^3 - rx = s.$

20. $x^4 - px^3 + rx = s.$

21. $x^4 - px^3 - rx = s.$

22. $-x^4 + px^3 + rx = s.$

23. $-x^4 + px^3 - rx = s.$

24. $-x^4 - px^3 + rx = s.$

25. $x^4 + px^3 + qx^2 = s.$

26. $x^4 + px^3 - qx^2 = s.$

27. $x^4 - px^3 + qx^2 = s.$

28. $x^4 - px^3 - qx^2 = s.$

29. $-x^4 + px^3 + qx^2 = s.$

30. $-x^4 + px^3 - qx^2 = s.$

31. $-x^4 - px^3 + qx^2 = s.$

32. $x^4 + px^3 + qx^2 + rx = s.$

33. $x^4 + px^3 + qx^2 - rx = s.$

34. $x^4 + px^3 - qx^2 + rx = s.$

35. $x^4 + px^3 - qx^2 - rx = s.$

36. $x^4 - px^3 + qx^2 + rx = s.$

37. $x^4 - px^3 + qx^2 - rx = s.$

38. $x^4 - px^3 - qx^2 + rx = s.$

39. $x^4 - px^3 - qx^2 - rx = s.$

40. $-x^4 + px^3 + qx^2 + rx = s.$

41. $-x^4 + px^3 + qx^2 - rx = s.$

42. $-x^4 + px^3 - qx^2 + rx = s.$

43. $-x^4 + px^3 - qx^2 - rx = s.$

44. $-x^4 - px^3 + qx^2 + rx = s.$

45. $-x^4 - px^3 + qx^2 - rx = s.$

46. $-x^4 - px^3 - qx^2 + rx = s.$

Art. CXXXV. Of theſe 46 forms of biquadratick equations, 27 forms, namely, all thoſe in which x^4 is not ſubtracted from the other terms on the left-hand ſide of the equation, are always poſſible, whatever may be the magnitudes of the co-efficients p, q, and r, and of the abſolute term s. Theſe forms are as follows:

1. $x^4 = s.$
2. $x^4 + rx = s.$
3. $x^4 - rx = s.$
4. $x^4 + qx^2 = s.$
5. $x^4 - qx^2 = s.$
6. $x^4 + px^3 = s.$
7. $x^4 - px^3 = s.$
8. $x^4 + qx^2 + rx = s.$
9. $x^4 + qx^2 - rx = s.$
10. $x^4 - qx^2 + rx = s.$
11. $x^4 - qx^2 - rx = s.$
12. $x^4 + px^3 + rx = s.$
13. $x^4 + px^3 - rx = s.$
14. $x^4 - px^3 + rx = s.$
15. $x^4 - px^3 - rx = s.$
16. $x^4 + px^3 + qx^2 = s.$
17. $x^4 + px^3 - qx^2 = s.$
18. $x^4 - px^3 + qx^2 = s.$
19. $x^4 - px^3 - qx^2 = s.$
20. $x^4 + px^3 + qx^2 + rx = s.$

21. $x^4 + px^3 + qx^2 - rx = s.$
22. $x^4 + px^3 - qx^2 + rx = s.$
23. $x^4 + px^3 - qx^2 - rx = s.$
24. $x^4 - px^3 + qx^2 + rx = s.$
25. $x^4 - px^3 + qx^2 - rx = s.$
26. $x^4 - px^3 - qx^2 + rx = s.$
27. $x^4 - px^3 - qx^2 - rx = s.$

The other 19 forms, in which x^4 is marked with the ſign —, or is ſubtracted from ſome of the other terms on the left-hand ſide of the equation, will ſometimes be impoſſible, if the abſolute term s is greater than the co-efficients p, q, and r, in a very great proportion.

Art. CXXXVI. And of the foregoing 27 forms of biquadratick equations, that are always poſſible, whatever be the magnitude of the abſolute term s, there are 20 forms that have but one root. Theſe forms are as follows:

1. $x^4 = s.$
2. $x^4 + rx = s.$
3. $x^4 - rx = s.$
4. $x^4 + qx^2 = s.$
5. $x^4 - qx^2 = s.$
6. $x^4 + px^3 = s.$
7. $x^4 - px^3 = s.$
8. $x^4 + qx^2 + rx = s.$
9. $x^4 + qx^2 - rx = s.$

10.

10. $x^4 - qx^2 - rx = s.$

11. $x^4 + px^3 + rx = s.$

12. $x^4 + px^3 - rx = s.$

13. $x^4 - px^3 - rx = s.$

14. $x^4 + px^3 + qx^2 = s.$

15. $x^4 + px^3 - qx^2 = s.$

16. $x^4 - px^3 - qx^2 = s.$

17. $x^4 + px^3 + qx^2 + rx = s.$

18. $x^4 + px^3 + qx^2 - rx = s.$

19. $x^4 + px^3 - qx^2 - rx = s.$

20. $x^4 - px^3 - qx^2 - rx = s.$

Art. CXXXVII. The remaining ſeven forms of the 27 forms ſet down in Art. 135, which are always poſſible, whatever be the magnitude of the abſolute term s, are theſe; to wit,

1. $x^4 - qx^2 + rx = s.$

2. $x^4 - px^3 + rx = s.$

3. $x^4 - px^3 + qx^2 = s.$

4. $x^4 + px^3 - qx^2 + rx = s.$

5. $x^4 - px^3 + qx^2 + rx = s.$

6. $x^4 - px^3 + qx^2 - rx = s.$

7. $x^4 - px^3 - qx^2 + rx = s.$

Theſe equations have ſometimes three roots, or, in the language of modern writers of Algebra, three real and poſitive, or affirmative, roots, if the co-efficients p, q, and r,

of x^3, x^2, and x, and the abſolute term s, are of certain proper relative magnitudes that are neceſſary for that purpoſe: but, when the abſolute term s is very great in compariſon of the ſaid co-efficients p, q, and r, they will have but one root. To determine when they will have three roots, and when they will have only one, and what will be the limits of the magnitudes of the leaſt and leaſt but one of the ſaid three roots, would be a very difficult and tedious buſineſs, on account of the great number of caſes into which the inquiry would be neceſſarily ſubdivided, ariſing from the variety of different proportions that may ſubſiſt between the four quantities p, q, r, and s. But the principle of the inquiry would be the Doctrine of *Maxima* and *Minima*, by which it would be our buſineſs to diſcover what would be the magnitude of the variable quantity x at the inſtant when the trinomial, or quadrinomial, quantity that forms the left-hand ſide of the equation in each of the ſaid ſeven forms, firſt becomes a *maximum*, or ceaſes to increaſe and begins to decreaſe, and what would be the ſubſequent magnitude of the ſame variable quantity x at the inſtant at which the ſaid trinomial, or quadrinomial, quantity, after having firſt increaſed to a certain magnitude, and begun to decreaſe from that magnitude, becomes a *minimum*, or ceaſes to decreaſe and begins to increaſe again. But of this I ſhall ſay nothing more on the preſent occaſion, on account of the prodigious length it would lead us to.

None of the foregoing ſeven forms of biquadratick equations will admit of more than three real and poſitive roots.

Art.

Art. CXXXVIII. The remaining 19 forms of biquadratick equations, which may ſometimes be impoſſible, when the abſolute term s exceeds the co-efficients p, q, and r in a very great degree, are as follows:

1. $-x^4 + rx = s$,
or $rx - x^4 = s$.
2. $-x^4 + qx^2 = s$,
or $qx^2 - x^4 = s$.
3. $-x^4 + px^3 = s$,
or $px^3 - x^4 = s$.
4. $-x^4 + qx^2 + rx = s$.
5. $-x^4 + qx^2 - rx = s$.
6. $-x^4 - qx^2 + rx = s$.
7. $-x^4 + px^3 + rx = s$.
8. $-x^4 + px^3 - rx = s$.
9. $-x^4 - px^3 + rx = s$.
10. $-x^4 + px^3 + qx^2 = s$.
11. $-x^4 + px^3 - qx^2 = s$.
12. $-x^4 - px^3 + qx^2 = s$.
13. $-x^4 + px^3 + qx^2 + rx = s$.
14. $-x^4 + px^3 + qx^2 - rx = s$.
15. $-x^4 + px^3 - qx^2 + rx = s$.
16. $-x^4 + px^3 - qx^2 - rx = s$.
17. $-x^4 - px^3 + qx^2 + rx = s$.
18. $-x^4 - px^3 + qx^2 - rx = s$.
19. $-x^4 - px^3 - qx^2 + rx = s$.

None of theſe 19 forms of biquadratick equations can have more than two real and poſitive roots, except the equation $-x^4 + px^3 - qx^2 + rx = s$; and that may have four roots. But, for the reaſon given in the preceding Article, I ſhall enter no further into the conſideration of the number of the roots of theſe equations, nor of the limits of their ſeveral magnitudes, but ſhall now proceed to ſhew how all the 46 forms of biquadratick equations ſet down in Art. 134 may be reſolved.

OF THE BIQUADRATICK EQUATION $x^4 = s$.

Art. CXXXIX. OF theſe equations the firſt and ſimpleſt is the equation $x^4 = s$. This equation may be reſolved by two ſucceſſive extractions of the ſquare-root of the known quantity, or abſolute term, s; the firſt of which extractions will give us the value of $\sqrt{s}$, or $s^{\frac{1}{2}}$, or xx, or the ſquare of the unknown quantity x; and the ſecond will give us that of $\sqrt{s^{\frac{1}{2}}}$, or of $\sqrt{xx}$, or of the unknown quantity x itſelf.

OF

OF THE BIQUADRATICK EQUATIONS, $x^4 + qx^2 = s$, $x^4 - qx^2 = s$, AND $qx^2 - x^4 = s$.

Art. CXL. NEXT to the pure equation $x^4 = s$, the 5th, 6th, and 7th equations ſet down in Art. 134, to wit, the equations $x^4 + qx^2 = s$, $x^4 - qx^2 = s$, and $-x^4 + qx^2 = s$, or $qx^2 - x^4 = s$, are the eaſieſt to be reſolved. For theſe equations have the form of quadratick equations, and may therefore be reſolved in the ſame manner as quadratick equations ſo far as to give us the value of xx; after which the value of x itſelf may be found by the extraction of the ſquare-root.

OF THE 11TH, 12TH, 13TH, 14TH, 15TH, 16TH AND 17TH FORMS OF BIQUADRATICK EQUATIONS SET DOWN IN ART. CXXXIV, IN WHICH FORMS THE CUBE OF THE UNKNOWN QUANTITY x IS WANTING.

Art. CXLI. OF the remaining 41 equations ſet down above in Art. 134, thoſe which ought next to be conſidered are the ſeven equations that come next after the 10th, namely, the 11th, 12th, 13th, 14th, 15th, 16th, and 17th, in which the ſecond term px^3, which involves the cube of

the unknown quantity, is wanting. Theſe equations are as follows:

1. $x^4 * + qx^2 + rx = s.$
2. $x^4 * + qx^2 - rx = s.$
3. $x^4 * - qx^2 + rx = s.$
4. $x^4 * - qx^2 - rx = s.$
5. $-x^4 * + qx^2 + rx = s.$
6. $-x^4 * + qx^2 - rx = s.$
7. $-x^4 * - qx^2 + rx = s.$

Theſe ſeven forms of biquadratick equations may be reſolved by the mediation of cubick equations by two different methods, of which one was invented by the famous *Monſieur Des Cartes*, and publiſhed in his Geometry in the year 1637; and the other was invented in the middle of the preceeding century, about the year 1545, by an ingenious gentleman of *Bologna* in Italy, whoſe name was *Lewis Ferrari*. Both their methods of reſolving theſe equations require a good deal of calculation, and conſequently are attended with a good deal of labour; and I think that in this reſpect they are nearly alike, and can ſee no reaſon for preferring one of them to the other. But with reſpect to another very important circumſtance, I mean the clearneſs of the ideas and reaſonings on which they are grounded, I think Ferrari's method greatly preferable to Des Cartes's; though (for I know not what reaſon,) it has been ſeldom mentioned by modern writers of Algebra, and Des Cartes's method has been generally adopted by them. Ferrari's method had, however, been duely honoured

noured by the writers of the century in which it was firſt publiſhed: for *Bombelli*, a learned Italian mathematician of that age, has employed the greater part of a very uſeful treatiſe of Algebra in explaining it.

OF THE BIQUADRATICK EQUATIONS

$x^4 - rx = s$, $x^4 - rx = s$, AND $-x^4 + rx = s$, OR $rx - x^4 = s$.

Art. CXLII. THE three equations $x^4 + rx = s$, $x^4 - rx = s$, and $rx - x^4 = s$, may be reſolved by either of the two methods mentioned in the foregoing Article as having been invented by Lewis Ferrari and Des Cartes, becauſe they may be conſidered as falling under the equations $x^4 + qx^2 + rx = s$ and $x^4 + qx^2 - rx = s$, and $-x^4 + qx^2 + rx = s$, when q, or the co-efficient of x^2, is ſuppoſed to be $= 0$. So that there now remain to be conſidered only ($46 - 1 - 3 - 7 - 3$, or $46 - 14$, or) 32 forms of biquadratick equations, which are thoſe in which the cube of the unknown quantity x is contained.

OF

OF THE REMAINING 32 FORMS OF BIQUADRATICK EQUATIONS, OR THOSE WHICH INVOLVE THE CUBE OF THE UNKNOWN QUANTITY.

Art. CXLIII. ALL the remaining biquadratick equations, (which are thoſe that involve x^3, or the cube of the unknown quantity x, and which are 32 in number,) may be reduced to equations of ſome one, or other, of the ſeven forms ſet down in Art. 141, by ſuppoſing x to be equal either to $y - \frac{p}{4}$, or to $\frac{p}{4} - y$, or to $\frac{p}{4} + y$, as the caſe ſhall require, and ſubſtituting the ſaid compound quantity inſtead of it in the terms of the propoſed equation; juſt as the cubick equations that involved the ſquare of the unknown quantity, ſet down in the foregoing part of this Appendix, were reduced to one, or other, of the three forms $y^3 + qy = r$, $y^3 - qy = r$, and $qy - y^3 = r$, by ſuppoſing x to be equal either to $y - \frac{p}{3}$, or to $\frac{p}{3} + y$, or to $\frac{p}{3} - y$. And conſequently all thoſe 32 forms of biquadratick equations may be ultimately reſolved (after having been thus reduced to ſome one of the ſaid ſeven forms,) by the mediation of cubick equations in the manner directed either by the method of Des Cartes, or by the clearer and more intelligible method of Lewis Ferrari.

Art. CXLIV. As to the reduction of the 32 biquadratick

tick equations that involve in them the ſecond term px^3, or the cube of the unknown quantity, to one of the ſeven forms ſet down in Art. 141, (in which the cube of the unknown quantity is wanting,) by ſuppoſing x to be equal to one of the three compound quantities $y - \frac{p}{4}$, $\frac{p}{4} - y$, and $\frac{p}{4} + y$, and ſubſtituting the ſaid compound quantities inſtead of x in the terms of the ſaid 32 equations, I ſhall not here enlarge upon it, but ſhall leave it intirely to the induſtry of the reader, becauſe the limits of this Appendix will not admit of ſo copious an addition to it as would be neceſſary to this diſcuſſion. And for the ſame reaſon I ſhall ſay nothing more of Des Cartes's method of reſolving biquadratick equations of the aforeſaid ſeven forms by the mediation of cubick equations, and ſhall alſo decline a full explanation of even Lewis Ferrari's method of doing the ſame thing, (though I very much prefer it to that of Des Cartes,) with reſpect to all the ſeven, or, rather, ten, equations to which it may be applied. But yet, as this method is both very ingenious and very clear and intelligible in the reaſonings on which it is grounded, I will juſt give the reader an idea of the nature of it, by applying it to the reſolution of an eaſy biquadratick equation that comes under the laſt of the ſeven forms of biquadratick equations ſet down above in Art. 141, to wit, the form $-x^4 - qx^2 + rx = s$, and which, (according to Mr. Montucla's account of the matter in his curious and valuable Hiſtory of the Mathematicks,) was derived from a certain arithmetical queſtion that had been propoſed to Lewis Ferrari, and the conſideration of which gave riſe to this moſt ingenious invention.

OF

OF THE RESOLUTION OF A BIQUADRATICK EQUATION OF THE LAST OF THE SEVEN FORMS SET DOWN ABOVE IN ART CXLI, TO WIT, THE EQUATION $-x^4 - 6xx + 60x = 36$, OR $60x - 6xx - x^4 = 36$, BY THE METHOD INVENTED BY SIGNIOR LEWIS FERRARI, OF BOLOGNA IN ITALY.

Art. CXLV. SIGNIOR LEWIS FERRARI was a native of Bologna in Italy, and a diſciple of the celebrated Cardan, a Phyſician and eminent Mathematician at Milan. Ferrari had an uncommon genius for the ſtudy of Algebra, which at that time had been recently improved by the two celebrated rules publiſhed by Cardan for reſolving certain cubick equations in which the ſquare of the unknown quantity was wanting. And the ability which he ſhewed in the proſecution of this ſtudy induced his teacher, Cardan, to have recourſe to him for his aſſiſtance towards the ſolution of a certain arithmetical Problem which had lately been propoſed to Cardan and the other Italian mathematicians of that time by a man named *John Colla*, and which none of them had been able to ſolve. This Problem was as follows: "*To find three numbers that were in continued proportion, or of which the firſt was to the ſecond as the ſecond is to the third, and that were of ſuch magnitudes that the ſum of all the three ſhould be equal to the number* 10, *and that the product ariſing from the multiplication of the firſt, or leaſt, by the ſecond, or middle number, ſhould*

ſhould be equal to the number 6." Some of the mathematicians to whom John Colla propoſed this Problem, conceived the ſolution of it to be impoſſible, or ſuppoſed that no numbers could be aſſigned that had the three foregoing properties. But Cardan was of a different opinion, and thought that there might be three numbers that had thoſe properties, though he was not able to find them. He therefore applied to his diſciple Lewis Ferrari (who was at that time a young man, and capable of intenſe application to ſubjects of this nature, as well as extremely fond of them,) and exhorted him to endeavour to find a ſolution of this Problem: and Lewis Ferrari yielded to his perſuaſions, and ſet about the buſineſs with great eagerneſs. In the purſuit of this ſolution he found that the Problem might be reduced to the reſolution of the following biquadratick equation, to wit, $36 + 6xx + x^4 = 60x$, or $60x - 6xx - x^4 = 36$, in which x repreſents the middle number of the three that he was endeavouring to find. And therefore his attention was then directed to the diſcovery of the roots of this equation. And in the proſecution of this inquiry he was ſo ſucceſsful as to find a method of diſcovering theſe roots which was capable of being applied to the reſolution of any other biquadratick equation whatſoever in which x^3, or the cube of the unknown quantity, is wanting, as well as to that of the particular biquadratick equation $60x - 6xx - x^4 = 36$, which he was then endeavouring to reſolve. The whole train of his reaſoning, both in firſt reducing the foregoing arithmetical Problem to the biquadratick equation $60x - 6xx - x^4 = 36$, and, afterwards, in his very ingenious method of inveſtigating the roots of that

that equation; I ſhall now proceed to lay before the reader it as full and clear a manner as I can.

THE REDUCTION OF THE FOREGOING ARITHMETICAL PROBLEM TO THE BIQUADRATICK EQUATION $60x - 6xx - x^4 = 36$.

Art. CXLVI. THE foregoing arithmetical Problem may be reduced to the biquadratick equation $60x - 6xx - x^4 = 36$, by proceeding as follows:

Let x be put for the ſecond of the three numbers which are in continued proportion, and of which the ſum is 10, and the product of the multiplication of the firſt, or leaſt, and the ſecond is 6.

Then it is evident that the firſt, or leaſt, of the ſaid three numbers will be $\frac{6}{x}$, and conſequently that the third, or greateſt, of them will be a third proportional to $\frac{6}{x}$ and x, and therefore will be equal to the ſquare of x divided by the fraction $\frac{6}{x}$, and conſequently to the ſquare of x multiplied into the reciprocal of that fraction, or into the fraction $\frac{x}{6}$, that is, to $xx \times \frac{x}{6}$, or to $\frac{x^3}{6}$. But, by another condition

condition of the Problem, the ſum of all the three numbers is equal to 10. Therefore we ſhall have $\frac{6}{x} + x + \frac{x^3}{6} = 10$; and conſequently (multiplying all the terms by 6,) we ſhall have $\frac{36}{x} + 6x + x^3 = 60$, and (multiplying all the terms by x,) $36 + 6xx + x^4 = 60x$, and (ſubtracting $6xx + x^4$ from both ſides,) $36 = 60x - 6xx - x^4$, or $60x - 6xx - x^4 = 36$.

Q. E. F.

THE RESOLUTION OF THE BIQUADRATICK EQUATION $60x - 6xx - x^4 = 36$, BY THE METHOD OF LEWIS FERRARI.

Art. CXLVII. IN order to reſolve the equation $60x - 6xx - x^4 = 36$, Lewis Ferrari proceeded as follows:

In the 1ſt place he added x^4 to both ſides of the equation; by which addition it became $60x - 6xx = 36 + x^4$. He then ſubtracted 36 from both ſides; by which it became $60x - 6xx - 36 = x^4$, or $x^4 = 60x - 6xx - 36$, in which the higheſt term x^4 ſtands by itſelf on the firſt, or left-hand, ſide of the equation.

Art.

Art. CXLVIII. Having ranged the terms in this order, Lewis Ferrari, in the next place, endeavoured to find ſuch a quantity as, being added to both ſides of the equation, would make both ſides of the new equation produced by ſuch addition be exact Algebräick ſquares, or be of ſuch magnitudes that each ſide of the ſaid new equation ſhould have a finite Algebräick quantity for it's ſquare-root; to the end that he might afterwards reduce the ſaid equation from a biquadratick equation to a quadratick equation by extracting the ſquare-roots of both it's ſides. And for this purpoſe he choſe the binomial quantity $2nxx + nn$, in which the letter n repreſents a quantity of which the magnitude is to be determined in a ſubſequent part of the Inveſtigation.

Now it is evident that this binomial quantity $2nxx + nn$ will anſwer the intended purpoſe with reſpect to x^4, or the left-hand ſide of the equation $x^4 = 60x - 6xx - 36$. For, if we add $2nxx + nn$ to x^4, the ſum will be $x^4 + 2nxx + nn$; of which the finite Algebräick expreſſion $xx + n$ is the exact ſquare-root. And this, it is evident, will be true, whatever be the magnitude of n; and conſequently we are at liberty, with reſpect to the left-hand ſide of the equation, to take n of any magnitude that we may think proper. If therefore it is poſſible to take n of ſuch a magnitude that, if the binomial quantity $2nxx + nn$ be added to the trinomial quantity $60x - 6xx - 36$, or the right-hand ſide of the equation $x^4 = 60x - 6xx - 36$, the ſum thereby produced, to wit, the quinquinomial quantity $60x - 6xx - 36 + 2nxx + nn$, or $2nx^2 - 6x^2 + 60x + nn - 36$, ſhall likewiſe be a perfect Algebräick ſquare, or have

have a finite Algebräick expreſſion for it's ſquare-root, the quantity $2nxx + nn$, when n ſhall be taken of that particular magnitude, will have the property required. We muſt therefore endeavour to find ſuch a value of n, (if ſuch a value can exiſt,) as will make the quinquinomial quantity $2nx^2 - 6x^2 + 60x + nn - 36$ be a perfect ſquare, or of ſuch a magnitude as to have a finite Algebräick expreſſion for it's ſquare-root.

Art. CXLIX. Now, ſince, whatever be the magnitude of n, the trinomial quantity $x^4 + 2nxx + nn$ will always be a perfect ſquare, of which the ſquare-root will be $xx + n$, let n be ſuppoſed to be of ſome magnitude greater than 6.

Then will nn be greater than 36, and $2nxx$ will be greater than $2 \times 6xx$, or than $12xx$, and, *à fortiori*, greater than $6xx$. Therefore 36 may be ſubtracted from nn, and $6x^2$ may be ſubtracted from $2nx^2$, and 6 may be ſubtracted from $2n$. Therefore $2nx^2 - 6x^2$ will be equal to $\overline{2n - 6} \times x^2$, and the binomial quantity $nn - 36$ may be conſidered as a ſingle quantity equal to the exceſs of nn above 36. And conſequently, if we put aa for $2n - 6$, and bb for $nn - 36$, the quinquinomial quantity $2nx^2 - 6x^2 + 60x + nn - 36$ will be equal to the trinomial quantity $aaxx + 60x + bb$. We muſt therefore inquire, whether it be poſſible for n to be taken of ſuch a magnitude, greater than 6, that the trinomial quantity $aaxx + 60x + bb$, or $\overline{2n - 6} \times xx + 60x + nn - 36$ ſhall be a perfect Algebräick ſquare, or ſhall have a finite Algebräick expreſſion for it's ſquare-root.

Art. CLII. Since, now that n is taken = 10.009,803, the trinomial quantity $aaxx + 60x + bb$ is = $aaxx + 2ab + bb$, it follows that $x^4 + 2nx^2 + nn$ (which, with any magnitude of n, greater than 6, is equal to $2n - 6 \times x^2 + 60x + nn - 36$, or to $aaxx + 60x + bb$,) will now be = $aaxx + 2abx + bb$. Therefore, by extracting the ſquare-roots of both ſides, we ſhall have $xx + n = ax + b$, that is, (becauſe n is = 10.009,803, and a is = 3.744,302, and b is = 8.012,250) xx + 10.009,803 will be = 3.744,302 × x + 8.012,250. And conſequently (ſubtracting 8.012,250 from both ſides,) we ſhall have xx + 1.997,553 = 3.744,302 × x, and, laſtly, (ſubtracting xx from both ſides,) we ſhall have 1.997,553 = 3.744,302 × $x - xx$, or 3.744,302 × $x - xx$ = 1.997,553. And thus we have at laſt, by the mediation of the cubick equation $n^3 - 3nn - 36n = 342$, reduced the original biquadratick equation $60x - 6xx - x^4 = 36$ to the quadratick equation 3.744,302 × $x - xx$ = 1.997,553. It now remains that we reſolve this quadratick equation.

Art. CLIII. Let both ſides of the equation 3.744,302 × $x - xx$ = 1.997,553 be ſubtracted from 3.504,949,366,801, or the ſquare of 1.872,151, or of half the number 3.744,302, which is the co-efficient of x. And we ſhall have 3.504,949,366,801 − 3.744,302 × $x + xx$ (= 3.504,949,366,801 − 1.997,553) = 1.507,396,366,801. Therefore the ſquare-root of the trinomial quantity 3.504,949,366,801 − 3.744,302 × $x + xx$, or of $1.872,151^2$ − 3.744,302 × $x + xx$, will be ($= \sqrt{1.507,396,366,801}$) = 1.227,760.

Now

Now this trinomial quantity has two ſquare-roots, of which the one is leſs than 1.872,151, or $\frac{3.744,302}{2}$, and the other is greater than 1.872,151, or $\frac{3.744,302}{2}$. The former of theſe roots is $1.872,151 - x$, and the latter is $x - 1.872,151$. Therefore we ſhall have, 1ſt, $1.872,151 - x = 1.227,760$, and conſequently $1.872,151 = 1.227,760 + x$, and $x (= 1.872,151 - 1.227,760) = 0.644,391$; and we ſhall have, 2dly, $x - 1.872,151 = 1.227,760$, and conſequently $x (= 1.227,760 + 1.872,151) = 3.099,911$. Therefore the two roots of the quadratick equation $3.744,302 \times x - xx = 1.997,553$ are 0.644,391 and 3.099,911; and conſequently the two roots of the biquadratick equation $60x - 6xx - x^4 = 36$, (from which the quadratick equation $3.744,302 \times x - xx = 1.997,553$ was derived,) will likewiſe be 0.644,391 and 3.099,911.

Q. E. I.

Art. CLIV. We will now ſubſtitute theſe two numbers 0.644,391 and 3.099,911 inſtead of x in the compound quantity $60x - 6xx - x^4$, which forms the left-hand ſide of the equation $60x - 6xx - x^4 = 36$, in order to ſhew that the values of the ſaid compound quantity reſulting from theſe ſubſtitutions are equal, or nearly equal, to 36, or the abſolute term of that equation, and conſequently that the ſaid numbers 0.644,391 and 3.099,911 are really the roots of the ſaid biquadratick equation.

Now, in the firſt place, if x be $= 0.644,391$, we ſhall have

$$xx (= \overline{0.644,391}^{2}) = 0.415,239,760,$$

and x^4 ($= \overline{0.644,391}^4$) $= 0.172,324,058$,

and conſequently $6xx$ ($= 6 \times 0.415,239,760$)

$= 2.491,438,560$,

and $60x$ ($= 60 \times 0.644,391$) $= 38.663,460$,

and $60x - 6xx - x^4$ ($= 38.663,460 - 2.491,438$,

$- 0.172,324 = 38.663,460$

$- 2.663,762) = 35.999,698$;

which differs from 36, or the abſolute term of the equation $60x - 6xx - x^4 = 36$, by only the very ſmall quantity 0.000,302, which is leſs than the 119,205th part of the ſaid abſolute term. Therefore 0.644,391 muſt be very nearly equal to the true value of the leſſer root of the biquadratick equation $60x - 6xx - x^4 = 36$.

Q. E. D.

And, ſecondly, if x is $= 3.099,911$, we ſhall have

xx ($= \overline{3.099,911}^2$) $= 9.609,448,207,921$,

and x^4 ($= \overline{3.099,911}^4$) $= 92.341,494,843$,

and conſequently $6xx$ ($= 6 \times 9.609,448,207$)

$= 57.656,689,242$,

and $60x$ ($= 60 \times 3.099,911$) $= 185.994,660$,

and $60x - 6xx - x^4$ ($= 185.994,660 - 57.656,689,242$,

$- 92.341,494,843 = 185.994,660$

$- 149.998,184) = 35.996,476$; which differs from 36, or the abſolute term of the equation $60x - 6xx - x^4 = 36$, by only the ſmall quantity 0.003,524, which is leſs than the 10,216th part of the ſaid abſolute term.

term. Therefore 3.099,911 must be very nearly equal to the true value of the greater root of the biquadratick equation $60x - 6xx - x^4 = 36$.

Q. E. D.

Art. CLV. We have now at last found, in the method invented by Lewis Ferrari, that the two roots of the biquadratick equation $60x - 6xx - x^4 = 36$ are 0.644,391 and 3.099,911. But of these roots the lesser has no relation to the arithmetical Problem above-mentioned, from which this equation was derived. For, since x was put for the middle number of the three which we were required to find, and the product of the multiplication of the said middle number and of the least number is supposed to be equal to 6, it follows that the said middle number must be greater than the square-root of 6, that is, than 2.449,490, &c, and therefore, *à fortiori*, greater than 0.644,391, or the lesser root of the equation $60x - 6xx - x^4 = 36$. Therefore the said middle number will be equal to the greater root of the said equation, to wit, 3.099,911. And consequently the least of the said three numbers, which we were required to find by John Colla's Arithmetical Problem, will be $(= \frac{6}{3.099,911}$, or) 1.935,539, and the greatest will be a third proportional to the first and second, that is, to 1.935,539 and 3.099,911, and consequently will be $(= \frac{3.099,911^2}{1.935,539} = \frac{9.609,448,207,921}{1.935,539})$ = 4.964,739; so that the three numbers required will be 1,935,539, 3.099,911, and 4.964.739.

Q. E. I.

The ſum of theſe three numbers, (which ought according to the conditions of the Problem to be equal to 10,) will be = 10.000,189, which exceeds the number 10 by only the ſmall quantity 0.000,189, which is leſs than the 52,910th part of 10. The three numbers 1.935,539 and 3.099,911 and 4.964,739 therefore anſwer the conditions of John Colla's Problem to a very conſiderable degree of exactneſs.

Q. E. D.

Art. CLVI. We have now gone compleatly through the reſolution of the biquadratick equation $60x - 6xx - x^4 = 36$ in the method invented by Lewis Ferrari, and have obtained the values of both it's roots to a conſiderable degree of exactneſs; but not without a great number of different proceſſes and operations, and a great deal of labour of calculation. And yet this is one of the ſimpleſt and eaſieſt biquadratick equations of one of the ſeven forms ſet down above in Art. 141, that could be pitched upon for an example of this method of reſolving equations of this kind. I am therefore inclined to think that it will be always moſt convenient in practice to reſolve equations of theſe forms, and, *à fortiori*, to reſolve all other affected biquadratick equations whatſoever (except the three equations, $x^4 + qx^2 = s$, $x^4 - qx^2 = s$, and $qx^2 - x^4 = s$, which may be reſolved in the manner of quadratick equations,) by Mr. Raphſon's method of approximation, which will be found much leſs laborious than the foregoing method of reſolving them. But that the difference of the two methods of reſolution in this important reſpect may be the more apparent, I ſhall now proceed to exhibit a reſolution of the foregoing

foregoing biquadratick equation $60x - 6xx - x^4 = 36$ by Mr. Raphſon's method of approximation.

A RESOLUTION OF THE FOREGOING BIQUADRATICK EQUATION $60x - 6xx - x^4 = 36$ BY MR. RAPHSON'S METHOD OF APPROXIMATION.

Art. CLVII. SINCE x is the ſecond of the three numbers ſought in the Arithmetical queſtion that gave riſe to this equation;—and, by one of the conditions of that queſtion, the product of the multiplication of the firſt, or leaſt, of thoſe numbers into the ſecond is equal to 6;—it follows that the ſaid ſecond number (which is greater than the firſt number, or other factor of the product 6,) muſt be greater than the ſquare-root of 6, that is, than 2.449, 490, &c. We will therefore ſuppoſe it to be nearly equal to 2.7, and try whether that number will anſwer the conditions of the Problem.

Now, if x is $= 2.7$, the firſt, or leaſt, number will be $\left(= \frac{6}{2.7}\right) = 2.222$, &c, and the third, or greateſt, number will be a third proportional to 2.222, &c, and 2.7, and conſequently will be $\left(= \frac{2.7 \times 2.7}{2.222, \&c} = \frac{7.29}{2.222, \&c}\right) = 3.280$ &c. But, by the laſt condition of the Problem, the ſum of the three numbers ſought is $= 10$. Therefore the ſum

ſum of the three numbers 2.222, &c, 2.7, and 3.280, &c ought, if thoſe numbers are nearly equal to the numbers ſought, to be nearly equal to 10. But the ſum of thoſe three numbers is equal only to 8.202 &c, which is much leſs than 10. Therefore the true value of x muſt be greater than 2.7.

We will therefore, in the next place, ſuppoſe it to be $= 2.9$, and try whether this number will anſwer the conditions of the Problem.

Now, if x is $= 2.9$, we ſhall have $\frac{6}{x} \left(= \frac{6}{2.9}\right) = 2.069$, and conſequently the third, or greateſt, number will be $\left(= \frac{2.9 \times 2.9}{2.069} = \frac{8.41}{2.069}\right) = 4.064$. Therefore the ſum of the three numbers ſought will, upon this ſecond ſuppoſition, be $(= 2.069 + 2.900 + 4.064) = 9.033$; which is ſtill too little. Therefore x muſt be greater than 2.9.

We will therefore, in the third place, ſuppoſe x to be $= 3$.

And then we ſhall have $\frac{6}{x} = \frac{6}{3}\Big) = 2$, and conſequently the third, or greateſt, number $\left(= \frac{3 \times 3}{2} = \frac{9}{2}\right) = 4.5$. Therefore the ſum of the three numbers ſought will, upon this ſuppoſition, be $(= 2 + 3 + 4.5) = 9.5$; which is ſtill too little, though the quantity by which it falls ſhort of the number 10 (to which it ought to be equal if the

the middle number x was exactly equal to 3,) is only 0.5. Therefore the true value of x muſt be a little greater than 3.

We will therefore, in the fourth place, ſuppoſe x to be $= 3.1$.

And then we ſhall have $\frac{6}{x}$ $(= \frac{6}{3.1})$ $= 1.9354$, and conſequently the third, or greateſt, number will be $(= \frac{3.1 \times 3.1}{1.9354} = \frac{9.61}{1.9354}) = 4.9653$. Therefore the ſum of the three numbers ſought will, upon this ſuppoſition, be $(= 1.9354 + 3.100 + 4.9653) = 10.0007$; which is a very little greater than 10. Therefore 3.1 muſt be a little greater than the true value of the middle number x, but will be ſo near to it as to make an excellent baſis for a further approximation to the ſaid true value by Mr. Raphſon's method. We will therefore now put an end to our conjectures and trials, and proceed by Mr. Raphſon's method.

Art. CLVIII. Let us therefore ſuppoſe x to be equal to $3.1 - z$, and ſubſtitute the ſaid compound quantity $3.1 - z$ inſtead of x in the terms of the propoſed equation $60x - 6xx - x^4 = 36$.

Then we ſhall have xx $(= \overline{3.1 - z}^2$
$= \overline{3.1}^2 - 2 \times 3.1 \times z +$ &c$) = 9.61 - 6.2 \times z +$ &c,
and x^4 $(= \overline{3.1 - z}^4 = \overline{3.1}^4 - 4 \times \overline{3.1}^3 \times z +$ &c

$=$

$$= 92.3521 - 4 \times 29.791 \times z + \&c)$$
$$= 92.3521 - 119.164 \times z + \&c,$$

and $60x$ $(= 60 \times \overline{3.1 - z} = 60 \times 3.1 - 60 \times z)$
$$= 186.0 - 60z,$$

and $6xx$ $(= 6 \times \overline{9.61 - 6.2 \times z + \&c} = 6 \times 9.61$
$$- 6 \times 6.2 \times z + \&c) = 57.66 - 37.2 \times z + \&c,$$

and consequently $60x - 6xx - x^4 =$

$$\left\{\begin{array}{l} 186.0000 - 60 \times z \\ -\ 57.6600 + 37.2 \times z \\ -\ 92.3521 + 119.164 \times z \end{array}\right\} = \left\{\begin{array}{l} 186.0000 - 60.0000 \times z \\ -150.0121 + 156.3640 \times z \end{array}\right\}$$

$$= 35.9879 + 96.3640 \times z.$$

But $60x - 6xx - x^4$ is $= 36$.

Therefore $35.9879 + 96.3640 \times z$ will also be $= 36$. And consequently $96.3640 \times z$ will be $(= 36.0000 - 35.9879) = 0.0121$, and z will be $\left(= \frac{0.0121}{96.3640}\right) = 0.000,125$. Therefore x, or $3.1 - z$, will be $(= 3.100,000 - 0.000,125) = 3.099,875$; that is, the greater of the two roots of the biquadratick equation $60x - 6xx - x^4 = 36$, or that which is equal to the second of the three numbers required to be found by the foregoing Arithmetical Problem, will be 3.099,875.

Q. E. I.

Art. CLIX. This value of x, to wit, 3.099,875, is somewhat more exact than the value of it found above, in Art. 153, by the method of Lewis Ferrari, which was 3.099,911.

3.099,911. For, if the middle number x be ſuppoſed to be = 3.099,875, the leaſt number will be $(= \frac{6}{3.099,875})$ = 1.935,561, and the greateſt number will be $(= \frac{3.099,875^2}{1.935,561} = \frac{9.609,225,015,625}{1.935,561})$ = 4.964,568; and conſequently the ſum of the three numbers ſought will be (= 1.935,561 + 3.099,875 + 4.964,568) = 9.999,904; which falls ſhort of the number 10 (to which it ought to be equal,) by the very ſmall fraction 0.000,096, which is leſs than the fraction 0.000,189, by which 10.000,189, or the ſum of the three numbers ſought, as obtained by the method of Lewis Ferrari in Art. 155, exceeds the ſaid number 10.

A COMPARISON BETWEEN THE PROCESSES USED IN THE FOREGOING RESOLUTION OF THE BIQUADRATICK EQUATION $60x - 6xx - x^4 = 36$ BY MR. RAPHSON'S METHOD OF APPROXIMATION, AND THE PROCESSES USED IN THE FORMER RESOLUTION OF THE SAME EQUATION BY THE METHOD OF LEWIS FERRARI IN ART. 147, 148, 149, 150, 151, 152, & 153.

Art. CLX. WE have ſeen, in the laſt article, that the value of x obtained by the reſolution of the biquadratick equation $60x - 6xx - x = 36$ by Mr. Raphſon's method of approximation, (though only one proceſs of the ſaid approximation was employed for the purpoſe,) to wit, the

the number 3.099,875, was ſomewhat more exact than the former value of x, to wit, 3.099,911, which was found above in Art. 147, - - - 153, by reſolving the ſame equation by the method of Lewis Ferrari. And yet the labour of finding the number 3.099,875 by Mr. Raphſon's method of approximation, has, it is evident, been much leſs than that of finding the number 3.099,911 by the method of Lewis Ferrari. But this will appear ſtill more clearly by recapitulating the ſeveral proceſſes which were uſed in thoſe different inveſtigations of the value of x.

Art. CLXI. In reſolving the equation $60x - 6xx - x^4 = 36$ by the method of Lewis Ferrari, our firſt ſtep was to arrange the terms of the equation in a new manner, ſo as to bring the term x^4 to be the only term on the firſt, or left-hand, ſide of the equation; whereby it was converted into the equation $x^4 = 60x - 6xx - 36$.

Our ſecond ſtep was to add to both ſides of this equation the binomial quantity $2nxx + nn$, in which n denoted a quantity of which the magnitude was to be determined afterwards. By this addition we obtained the equation $x^4 + 2nxx + nn = 60x - 6xx - 36 + 2nxx + nn$; of which equation the firſt, or left-hand, ſide, to wit, the trinomial quantity $x^4 + 2nxx + nn$, is a perfect Algebraick ſquare, the ſquare-root of which is the binomial quantity $xx + n$. And this, we obſerved, muſt always be true, whatever be the magnitude of the aſſumed number n.

We then, in the third place, proceeded to conſider what was the lower limit of the magnitude that ought to be given to the number n, to the end that it might be poſſible for the quinquinomial quantity $60x - 60xx -$

36

$36 + 2nxx + nn$ to be a perfect Algebräick ſquare; and we found that it was neceſſary that the ſaid number ſhould be of ſome magnitude greater than 6. We therefore ſuppoſed it to be of ſome magnitude greater than 6. And hence it followed that $6xx$ would be leſs than $2nxx$, and that 36 would be leſs than nn, and that the quinquinomial quantity $6xx - 36 + 2nxx + nn$, (which formed the ſecond, or right-hand, ſide of the laſt equation $x^4 + 2nxx + nn = 60x - 6xx - 36 + 2nxx + nn$) would then become equal to the trinomial quantity $60x + \overline{2n-6} \times xx + \overline{nn-36}$, or $\overline{2n-6} \times xx + 60x + nn - 36$, or, if we put $aa = 2n - 6$, and $bb = nn - 36$, to the trinomial quantity $aaxx + 60x + bb$, and therefore that the laſt equation $x^4 + 2nxx + nn = 60x - 6xx - 36 + 2nxx + nn$ would thereby be converted into the equation $x^4 + 2nxx + nn = aaxx + 60x + bb$, in which the trinomial quantity $aaxx + 60x + bb$ is of ſuch a form as to be capable of becoming a perfect Algebräick ſquare, of which the binomial quantity $ax + b$ will be the ſquare-root, provided the quantities aa and bb are of certain proper magnitudes with reſpect to the number 60.

We then, in the fourth place, proceeded to demonſtrate that we were at liberty to take n of any magnitude, greater than 6, that we thought fit, and to ſhew that, if we took it of ſuch a magnitude as to make it equal to the root of the cubick equation $n^3 - 3nn - 36n = 342$, the quantity $60x$ would be equal to $2abx$, or to $2 \times \sqrt{\overline{2n-6}} \times \sqrt{\overline{nn-36}}$, and conſequently that the quantity $aaxx + 60x + bb$ would be equal to $aaxx + 2abx + bb$.

And

And then, in the fifth place, we reſolved the cubick equation $n^3 - 6nn - 36n = 342$, and found it's root to be 10.009,803; which we therefore choſe for the magnitude to be aſſigned to the quantity n in the equation $x^4 + 2nxx + nn = aaxx + 60x + bb$, and thereby (becauſe $2abx$ is in this caſe $= 60x$) we converted the ſaid equation into the equation $x^4 + 2nxx + nn = aaxx + 2abx + bb$, in which there was then no unknown quantity but x, and of which the right-hand ſide $aaxx + 2abx + bb$ is a perfect Algebräick ſquare, to wit, the ſquare of the binomial quantity $ax + b$, as well as the left-hand of it, or the trinomial quantity $x^4 + 2nxx + nn$, which we had before obſerved to be the ſquare of the binomial quantity $xx + n$.

We then, in the ſixth place, extracted the ſquare-roots of the two trinomial quantities $x^4 + 2nxx + nn$ and $aaxx + 2abx + bb$, which form the two oppoſite ſides of the equation $x^4 + 2nxx + nn = aaxx + 2ab + bb$; and we thereby obtained the quadratick equation $xx + n = ax + b$, or $xx + 10.009,803 = ax + b$, or, becauſe aa is $(= 2n - 6 = 2 \times 10\,009,803 - 6 = 20.019,606 - 6) = 14.019,606$, and bb is $(= nn - 36 = \overline{10.009,803}^2 - 36 = 100.196,156,098,809 - 36) = 64.196,156,098,809$, the quadratick equation $xx + 10.009,803 = \sqrt{14.019,606} \times x + \sqrt{64.196,156,098,809}$, or (becauſe $\sqrt{14.019,606}$ is $= 3.744,302$, and $\sqrt{64.196,156,098,809}$ is $= 8,012,250$,) the quadratick equation $xx + 10.009,803 = 3.744,302 \times x + 8.012,250$. The ſquaring the number n, or 10.009,803, and the extracting the ſquare-roots of the two numbers aa and bb, or 14.019,606 and 64.196,156,

156,098,809, (which are neceſſary operations to the compleat attainment of the quadratick equation $xx + n = ax + b$, or $xx + 10.009,803 = 3.744,302 \times x + 8.012,250$,) are operations that make this ſixth ſtage of our inveſtigation rather tedious and troubleſome.

In the ſeventh place we reduced the quadratick equation $xx + 10.009,803 = 3.744,302 \times x + 8.012,250$ into a proper form for being reſolved; by firſt ſubtracting 8.012,250 from both ſides of it, which produced the equation $xx + 1.997,553 = 3.744,302 \times x$; and then ſubtracting xx from both ſides, which produced the equation $1.997,553 = 3.744,302 \times x - xx$, or $3.744,302 \times x - xx = 1.997,553$.

And, eighthly, and laſtly, we reſolved the quadratick equation $3.744,302 \times x - xx = 1.997,553$, and found it's two roots to be 0.644,391 and 3.099,911. And in performing this reſolution it was neceſſary to ſquare the number 1.872,151, (which is equal to $\frac{3.744,302}{2}$) and likewiſe to extract the ſquare-root of the number 1.507,396,366,801, which was found to be 1.227,760.

Now all theſe eight proceſſes of this reſolution of the biquadratick equation $60x - 6xx - x^4 = 36$ by the method of Lewis Ferrari, form a very conſiderable body both of cloſe and cautious reaſoning and of troubleſome arithmetical calculation.

Art. CLXII. Now to theſe proceſſes we are to compare the proceſſes uſed in reſolving the ſame equation $60x - 6xx - x^4 = 36$ in Art. 157 and 158, by Mr. Raphſon's method of approximation. Theſe proceſſes were as follows:

The firſt ſtep of this reſolution was to obſerve, that, ſince the product of the multiplication of the leaſt of the three numbers ſought by the middle number x was equal to the number 6, the ſaid middle number x muſt be greater than the ſquare-root of 6, or than 2.449,490, &c, and, in conſequence of this obſervation, to conjecture that it might be equal to 2.7, and to try the effect of ſuch a ſuppoſition. The reſult of this trial was, that, if x was ſuppoſed to be equal to 2.7, the leaſt number would be $\left(= \frac{6}{2.7}\right)$ = 2.222, &c, and the greateſt number would be $\left(= \frac{2.7 \times 2.7}{2.222, \&c} = \frac{7.29}{2.222, \&c}\right) = 3.280$ &c, and conſequently that the ſum of the three numbers ſought would be = 8.202, &c, inſtead of being = 10, as it ought to be by the conditions of the Problem. And hence we concluded that the ſaid middle number would be greater than 2.7.

We then, in the ſecond place, ſuppoſed the ſaid middle number x to be = 2.9, and tried the effects of that ſuppoſition. And the reſult was, that, upon this ſuppoſition, the leaſt number would be $\left(= \frac{6}{2.9}\right) = 2.069$, and the greateſt number would be $\left(= \frac{2.9 \times 2.9}{2.069} = \frac{8.41}{2.069}\right) =$

4.064

4.064, and consequently that the sum of the three numbers sought would be 9.033; which is less than 10, to which, by the conditions of the Problem, the said sum ought to be equal. We therefore concluded that the said middle number must also be greater than 2.9.

We then, in the third place, supposed the said middle number to be = 3, and tried the effects of that supposition. And the result was, that, upon this supposition, the least number would be $(= \frac{6}{3}) = 2$, and the greatest number would be $(= \frac{3 \times 3}{2} = \frac{9}{2}) = 4.5$, and consequently that the sum of the three numbers sought would be = 9.5; which still, like the former results, is less than 10, but differs from it by only 0.5. We therefore concluded that the said middle number must also be greater than 3, but that their difference would be but small.

We then, in the fourth place, supposed the middle number x to be = 3.1, and tried the effects of that supposition. And the result was, that, upon this supposition, the least number would be $(= \frac{6}{3.1}) = 1.9354$, and the greatest would be $(= \frac{3.1 \times 3.1}{1.9354} = \frac{9.61}{1.9354}) = 4.9653$, and consequently that the sum of all the three numbers would be = 10.0007; which is a very little greater than 10, to which, by the conditions of the Problem, it ought to be equal. We therefore concluded that 3.1 must be greater than the said middle number x, but that it would exceed it by a very

ſmall quantity, and therefore that the ſaid number 3.1 would be an excellent baſis on which to found a further approach to the true value of the ſaid middle number x, or to the root of the biquadratick equation $60x - 6xx - x^4 = 36$, by Mr. Raphſon's method.

We then, in the fifth place, ſuppoſed x to be $= 3.1 - z$, and ſubſtituted the ſaid compound quantity $3.1 - z$ inſtead of x in the terms of the equation $60x - 6xx - x^4 = 36$, and reſolved the new equation reſulting from ſuch ſubſtitution in the manner directed by Mr. Raphſon. And the reſult was, that z was $= 0.000,125$, and conſequently that $3.1 - z$, or the ſaid middle number x, would be $(= 3.100,000 - 0.000,125) = 3.099,875$.

Art. CLXIII. Now theſe conjectures and trials of the four ſmall numbers 2.7, 2 9, 3, and 3.1, as probable near values of the middle number x of the three numbers ſought in the above-mentioned Arithmetical Problem, together with the ſingle proceſs of Mr. Raphſon's method of approximation by ſubſtituting $3.1 - z$ inſtead of x in the terms of the equation $60x - 6xx - x^4 = 36$, and reſolving the new equation thereby obtained, ſeem to be much leſs difficult and laborious in point of calculation, and likewiſe leſs ſubtle and abſtruſe in the reaſonings employed in them, than the eight different ſtages, or proceſſes, of the former reſolution of the ſame equation $60x - 6xx - x^4 = 36$ by the method of Lewis Ferrari. And therefore I think we may now fairly conclude that Mr. Raphſon's method of reſolving biquadratick equations by approximation is very much to be preferred, in practice at leaſt,

leaſt, to the method of Lewis Ferrari with reſpect to all biquadratick equations of the ſeven forms ſet down above in Art. 141, and alſo of the three forms ſet down in Art. 142, that is, with reſpect to all ſuch biquadratick equations as are without the term px^3, or the cube of the unknown quantity.

Art. CLXIV. And, if the aforeſaid method of reſolving equations of this kind by approximation according to the directions of Mr. Raphſon deſerves to be preferred to the other method of reſolving the ſame equations invented by Lewis Ferrari, it ought, *à fortiori*, to be preferred to the obſcure and ſubtle method of reſolving equations of the ſame kind that has been given us by Monſieur Des Cartes in his Geometry. And therefore I think it may be confidently affirmed that, whenever we have occaſion to reſolve a biquadratick equation of any of the aforeſaid forms, or in which the cube of the unknown quantity is not contained, the beſt method that can be taken for that purpoſe will be to find, at firſt, a tolerably near value of the root ſought by a few conjectures and trials, made with ſmall numbers, conſiſting only of one or two figures each, and afterwards to make a nearer approach to the true value of ſuch root by one, or two, or more, proceſſes of Mr. Raphſon's method of approximation.

Art. CLXV. And, if the ſaid method of reſolving equations by approximation in the manner taught by Mr. Raphſon deſerves to be preferred to the methods of reſolving biquadratick equations invented by Lewis Ferrari

and Monſieur Des Cartes with reſpect to ſuch of thoſe equations as are without the cube of the unknown quantity, it muſt, *à fortiori*, be preferable to thoſe methods of reſolving biquadratick equations of other kinds, or that involve in them the cube of the unknown quantity; becauſe thoſe other biquadratick equations cannot be reſolved by the methods of Lewis Ferrari and of Des Cartes till they have been, firſt, reduced, by ſome of the ſubſtitutions mentioned above in Art. 143, to ſome of the ſeven forms mentioned in Art. 141, in which the cube of the unknown quantity is wanting; and theſe reductions require a good deal of calculation, and much increaſe the length and trouble of reſolving them by either of thoſe methods. Therefore we may conclude that, in the buſineſs of reſolving biquadratick equations, it will always be more convenient to have recourſe to Mr. Raphſon's method of reſolving them by approximation, than to attempt the reſolution of them by the methods of either Lewis Ferrari or Des Cartes, whether the ſaid equations do, or do not, involve in them the cube of the unknown quantity; excepting only the three equations $x^4 + qx^2 = s$, $x^4 - qx^2 = s$, and $qx^2 - x^4 = s$, which may be reſolved in the manner of quadratick equations.

Art. CLXVI. It is true indeed that the principles of Lewis Ferrari's method of reſolving biquadratick equations in which the cube of the unknown quantity is wanting, have been extended by Dr. Waring (the learned Lucaſian Profeſſor of Mathematicks in the Univerſity of Cambridge,) in his *Meditationes Algebraicæ*, page 92, and likewiſe by the late eminent Algebraiſt, Mr Thomas Simpſon of Woolwich

Woolwich Academy, in the 6th edition of his Algebra, page 150, (and, I believe, in ſome of the former editions, but not in the firſt edition of it,) to biquadratick equations that have all their terms compleat. But the labour of the calculations that are required in ſuch a method of proceeding is ſtill greater than that of, firſt, reducing a biquadratick equation of that kind to one of the ſeven forms ſet down in Art. 141, and afterwards reſolving the reduced equation by the rules of Lewis Ferrari's method. I have tried both theſe methods of reſolving thoſe compleat equations, in the caſe of a biquadratick equation that not only had all it's terms compleat, but which had four real and affirmative roots; which is the moſt intricate and difficult form of biquadratick equations that can be conſidered: and therefore I can ſpeak poſitively on the ſubject. And, from the experience I have derived from theſe trials, I will venture to recommend it to my readers, whenever they have occaſion to reſolve a biquadratick equation of any kind (except of the three forms above-excepted, which may be conſidered as quadratick equations,) to have recourſe at once to Mr. Raphſon's method of reſolving them by approximation.

Art. CLXVII. Before I quit this ſubject of biquadratick equations, I will give one more inſtance of the reſolution of one of theſe equations by the foregoing method of Lewis Ferrari; and that ſhall be in the caſe of an equation that involves in it the ſecond term px^3, or the cube of the unknown quantity, and therefore will afford an example of the reduction of one of thoſe biquadratick equations to one of the ſeven forms ſet down above in Art. 141, before

the above-deſcribed proceſſes of Lewis Ferrari's method of reſolution can be applied to it. And the equation I ſhall chuſe for this purpoſe ſhall be one that may be derived from the ſame Arithmetical Problem, propoſed by *John Colla*, from which we derived the foregoing biquadratick equation $60x - 6xx - x^4 = 36$ which has been the ſubject of the foregoing articles. We will therefore now reſume the conſideration of the ſaid Arithmetical Problem, and give another ſolution of it.

ANOTHER SOLUTION OF THE ABOVE-MENTIONED ARITHMETICAL PROBLEM, THAT WAS PROPOSED BY JOHN COLLA, AND PRODUCED THE FOREGOING BIQUADRATICK EQUATION $60x - 6xx - x^4 = 36$.

Art. CLXVIII. LET the leaſt of the three numbers that are in continued proportion be denoted by the letter x, inſtead of the ſecond of thoſe numbers which was ſo denoted in the former ſolution of this Problem.

Then, ſince the product of the multiplication of the firſt, or leaſt, number into the ſecond number is, by one of the conditions of the Problem, equal to 6, it follows that the ſecond number muſt be $= \frac{6}{x}$.

But, by another condition of the Problem, the third, or greateſt, number is a third proportional to the firſt and ſecond.

cond. Therefore the ſaid greateſt number muſt be equal to the ſquare of the ſecond number $\frac{6}{x}$ divided by the firſt number x, and conſequently muſt be $= \frac{\frac{36}{xx}}{x} = \frac{36}{x^3}$.

But, by the third condition of the Problem, the ſum of theſe three numbers will be equal to 10. Therefore we ſhall have $x + \frac{6}{x} + \frac{36}{x^3} = 10$; and conſequently (multiplying all the terms by x^3,) $x^4 + 6xx + 36 = 10x^3$, and (ſubtracting $x^4 + 6xx$ from both ſides,) $10x^3 - x^4 - 6xx = 36$, or $-x^4 + 10x^3 - 6xx = 36$. Therefore, in order to ſolve the foregoing Arithmetical Problem, it will be neceſſary that we ſhould reſolve the biquadratick equation $10x^3 - x^4 - 6xx = 36$, or $-x^4 + 10x^3 - 6xx = 36$, which involves in it the cube of the unknown quantity x.

OF THE RESOLUTION OF THE BIQUADRATICK EQUATION $10x^3 - x^4 - 6xx = 36$ BY THE METHOD OF LEWIS FERRARI.

Art. CLXIX. IN this equation $10x^3 - x^4 - 6xx = 36$ the value of x muſt be leſs than $\frac{10}{4}$, or 2.5. For, ſince the product of the multiplication of the firſt and ſecond of the three numbers which we are ſeeking is only equal

equal to 6, it is evident that x, or the leſſer of thoſe two numbers muſt be leſs than the ſquare-root of 6, or than 2.449,490, &c, and conſequently, *à fortiori*, leſs than 2.5, which is greater than 2.449,490, &c. It will therefore be equal to the exceſs of 2.5, or $\frac{10}{4}$, above ſome other quantity, which we will call y, or to the compound quantity $2.5 - y$.

Now let this quantity $2.5 - y$ be ſubſtituted inſtead of x in the equation $10x^3 - x^4 - 6xx = 36$. This may be done in the manner following:

Since x is $= 2.5 - y$, we ſhall have

$xx\ (= \overline{2.5-y}^2 = \overline{2.5}^2 - 2 \times 2.5 \times y + yy) = 6.25 - 5y + yy$,

and $x^3\ (= \overline{2.5-y}^3 = \overline{2.5}^3 - 3 \times \overline{2.5}^2 \times y + 3 \times 2.5 \times yy - y^3$
$= \overline{2.5}^3 - 3 \times 6.25 \times y + 7.5yy - y^3) = 15.625 - 18.75y$
$+ 7.5yy - y^3$,

and $x^4\ (= \overline{2.5-y}^4 = \overline{2.5}^4 - 4 \times \overline{2.5}^3 \times y + 6 \times \overline{2.5}^2 \times yy$
$- 4 \times 2.5 \times y^3 + y^4 = \overline{2.5}^4 - 4 \times 15.625 \times y + 6 \times 6.25$
$\times yy - 10.0 \times y^3 + y^4) = 39.0625 - 62.500 \times y +$
$37.50 \times yy - 10.0 \times y^3 + y^4$.

Therefore $10x^3$ will be $(= 10 \times \overline{15.625 - 18.75y + 7.5yy - y^3}$
$= 10 \times 15.625 - 10 \times 18.75y + 10 \times 7.5yy - 10 \times y^3)$
$= 156.25 - 187.5y + 75yy - 10y^3$,

and $6xx$ will be $(= 6 \times \overline{6.25 - 5y + yy} = 6 \times 6.25 - 6$
$\times 5y + 6 \times yy) = 37.50 - 30y + 6yy$,

and

and consequently $10x^3 - 6xx - x^4$ will be =

$$\left\{\begin{array}{l} 156.25 - 187.5 \times y + 75yy - 10y^3 \\ -37.50 + 30.0 \times y - 6yy \\ -39.0625 + 62.500y - 37.50yy + 10y^3 - y^4 \end{array}\right\}$$

$= 79.6875 - 95.00y + 31.50yy * - y^4.$

But $10x^3 - 6xx - x^4$ is $= 36.$

Therefore $79.6875 - 95.00y + 31.50yy - y^4$ will also be $= 36$. And consequently (adding $95.00y + y^4$ to both sides,) $79.6875 + 31.50yy$ will be $= 36 + 95.00y + y^4$, and (subtracting $31.50yy$ from both sides,) 79.6875 will be $= 36 + 95.00y - 31.50yy + y^4$, and lastly, (subtracting 36 from both sides,) 43.6875 will be $= 95.00y - 31.50yy + y^4$, or $y^4 - 31.50yy + 95.00y$ will be $= 43.6875$, or $y^4 - 31.50yy + 95y$ will be $= 43.6875$. And thus we have now converted the first equation $10x^3 - 6xx - x^4 = 36$, which involved in it the cube of the unknown quantity x, into the equation $y^4 - 31.50yy + 95y = 43.6875$, in which the cube of the unknown quantity y is wanting, and which therefore is capable of being resolved by the method of Lewis Ferrari.

Art. CLXX. To resolve this equation $y^4 - 31.50yy + 95y = 43.6875$ by the method of Lewis Ferrari, we must add $31.50yy$ to both sides of the equation, by which we shall obtain the equation $y^4 + 95y = 43.6875 + 31.50yy$; and we must then subtract $95y$ from both sides, by which means we shall obtain the equation $y^4 = 43.6875 + 31.50yy - 95y$, or $y^4 = 31.50yy - 95y + 43.6875$,

6875, in which the term y^4 ſtands by itſelf on the left-hand ſide of the equation, as it ought to do in order to prepare the equation for the ſubſequent operations of this method of reſolution.

In the next place we muſt add the binomial quantity $2nyy + nn$ to both ſides of the laſt equation $y^4 = 31.50yy - 95y + 43.6875$. And we ſhall thereby obtain the equation $y^4 + 2nyy + nn = 31.50yy - 95y + 43.6875 + 2nyy + nn$, in which the trinomial quantity $y^4 + 2nyy + nn$, that is on left-hand ſide of the mark of equality $=$, is a perfect Algebräick ſquare, of which the binomial quantity $yy + n$ is the ſquare-root. The value of the aſſumed quantity n remains ſtill to be determined: but, whatever it's value is, the binomial quantity $yy + n$ will equally be the exact ſquare-root of the trinomial quantity $y^4 + 2nyy + nn$.

In the third place, let aa be put $= 2n + 31.50$, and bb be put $= nn + 43.6875$.

Then will the quinquinomial quantity $31.50yy - 95y + 43.6875 + 2nyy + nn$ (which is equal to $2n + 31.50 \times yy - 95y + nn + 43.6875$,) be $= aayy - 95y + bb$; and conſequently the equation $y^4 + 2nyy + nn = 31.50yy - 95y + 43.6875 + 2nyy + nn$ will become $y^4 + 2nyy + nn = aayy - 95y + bb$.

In the fourth place we muſt endeavour to take the quantity n of ſuch a magnitude (if poſſible,) that $2ab$ ſhall be equal to 95. For, if that be once effected, we ſhall have

have $y^4 + 2nyy + nn = aayy - 2aby + bb$; in which equation the trinomial quantity $aayy - 2aby + bb$, which forms the ſecond, or right-hand, ſide of the equation, is a perfect Algebräick ſquare, (to wit, the ſquare of the reſidual quantity $ay - b$, or of the reſidual quantity $b - ay$,) as well as the trinomial quantity $y^4 + 2nyy + nn$, which forms the left-hand ſide of the equation: and conſequently, if this can be done, we ſhall be able to reduce the equation from a biquadratick equation to a quadratick equation by extracting the ſquare-roots of both it's ſides.

Art. CLXXI. Now $2ab$ will be $= 95$, when ab is $= \frac{95}{2}$, and $aabb$ is $= \overline{\frac{95}{2}}^2$. But $aabb$ is $= \overline{2n + 31.5c} \times \overline{nn + 43.6875} = 2n^3 + 31.5nn + 87.3750n + 1376.15625$. Therefore $2ab$ will be $= 95$, when $2n^3 + 31.5nn + 87.3750n + 1376.15625$ is equal to $\overline{\frac{95}{2}}^2$, or to $\frac{\overline{95}^2}{4}$, or to $\frac{9025}{4}$, or to 2256.25, or when $n^3 + 15.75nn + 43.6875n + 688.078,125$ is $= 1128.125$, or when $n^3 + 15.75nn + 43.6875n$ is $= (1128.125 - 688.078,125$, or) $440.046,875$. Now this is evidently a poſſible equation; becauſe, if we ſuppoſe n to increaſe gradually from 0 *ad infinitum*, each of the three quantities n^3, $15.75nn$, and $43.6875n$ will at the ſame time increaſe gradually from 0 *ad infinitum*, and conſequently their ſum, or the trinomial quantity $n^3 + 15.75nn + 43.6875n$, will alſo increaſe gradually from 0 *ad infinitum*, and conſequently will, during it's increaſe, become ſucceſſively equal to all quantities whatſoever, how ſmall or how great ſoever; and therefore

fore it will, at ſome inſtant of time during it's ſaid increaſe, become equal to the quantity 440.046,875. We muſt therefore now reſolve the cubick equation $n^3 + 15.75nn + 43.6875n = 440.046,875$, and take the number n equal to the root of this equation; and then we ſhall have $2aby = 95y$, and $y^4 + 2nyy + nn$ (which has been ſhewn to be equal to $aayy - 95y + bb$) will be $= aayy - 2aby + bb$. Now this equation $n^3 + 15.75nn + 43.6875n = 440.046,875$ has been reſolved above in Art. 131 and 132, and it's root has been found to be nearly equal to, but ſomewhat leſs than, 3.760. We muſt therefore take $n = 3.760$, and then we ſhall have $y^4 + 2nyy + nn = aayy - 2aby + bb$, and conſequently, (extracting the ſquare-roots on both ſides,) $yy + n = ay - b$, or $b - ay$, according as ay is greater or leſs than b, or $yy + 3.760 = ay - b$, or $b - ay$, according as ay is greater or leſs than b.

Art. CLXXII. We muſt next compute the values of a and b.

Now, ſince aa is $= 2n + 31.50$, and bb is $= nn + 43.6875$, and n is $= 3.760$, we ſhall have aa $(= 2 \times 3.760 + 31.50 = 7.520 + 31.50) = 39.020$, and bb $(= 3.760^2 + 43.6875 = 14.1376 + 43.6875) = 57.8251$, and conſequently a $(= \sqrt{39.020}) = .6.246$, and b $(= \sqrt{57.8251}) = 7.604$. Therefore the laſt equation $yy + 3.760 = ay - b$, or $b - ay$, according as ay is greater, or leſs, than b, will now be converted into the equation $yy + 3.760 = 6.246 \times y - 7.604$, or $7.604 - 6.246 + y$, according as $6.246 \times y$ is greater or leſs than 7.604. We muſt therefore

therefore inquire, whether $6.246 \times y$ is greater or leſs than 7.604.

Art. CLXXIII. Now $6.246 \times y$ will be greater, or leſs, than 7.604, according as y is greater, or leſs, than $\frac{7.604}{6.246}$, or 1.217. We muſt therefore inquire, whether y is greater, or leſs, than 1.217.

Now, if y were $= 1.217$, we ſhould have $yy = 1.481,089$, and $y^4 = 2.193,361$, and conſequently $31.5yy$ $(= 31.5 \times 1.481,089) = 46.654,303$, and $95y$ $(= 95 \times 1.217) = 115.615$, and $y^4 - 31.5yy + 95y$ $(= 2.193,361 - 46.654,303 + 115.615 = 117.808,361 - 46.654,303) = 71.154,058$. But this number is much greater than the number 43.6875, or the abſolute term of the equation $y^4 - 31.5yy + 95y = 43.6875$. Conſequently 1.217 is greater than the true value of y in that equation. Therefore $6.246 \times y$ muſt be leſs than 7.604, or ay muſt be leſs than b. Therefore the ſquare-root of the trinomial quantity $aay - 2aby + bb$ will be $b - ay$, and not $ay - b$.

Q. E. I.

We ſhall therefore have $yy + n = b - ay$, or $yy + 3.760 = 7.604 - 6.246 \times y$. And conſequently, (adding $6.246 \times y$ to both ſides,) we ſhall have $yy + 6.246 \times y + 3.760 = 7.604$, and (ſubtracting 3.760 from both ſides,) $yy + 6.246 \times y = 3.844$. And thus we have reduced the biquadratick equation $y^4 - 31.5yy + 95y = 43.6875$ to the quadratick equation $yy + 6.246 \times y$

$=$

$= 3.844$. This quadratick equation we muſt now reſolve.

Art. CLXXIV. To both ſides of the quadratick equation $yy + 6.246 \times y = 3.844$, add $\overline{3.123}^2$, or the ſquare of half the co-efficient 6.246 of y in the ſaid equation. And we ſhall have $yy + 6.246 \times y + \overline{3.123}^2 (= 3.844 + \overline{3.123}^2 = 3.844 + 9.753,129) = 13.597,129$. Therefore (extracting the ſquare-roots of both ſides,) we ſhall have $y + 3.123 = 3.687$, and (ſubtracting 3.123 from both ſides) $y = 0.564$. Therefore the root of the biquadratick equation $y^4 - 31.5yy + 95y = 43.6875$ will be $= 0.564$.

Q. E. I.

Therefore x, or $\frac{10}{4} - y$, or $2.500 - y$, will be $(= 2.500 - 0.564) = 1.936$; that is, the root of the firſt equation $10x^3 - 6xx - x^4 = 36$, or the leaſt of the three numbers in continued proportion which, in the Arithmetical Problem above-mentioned, we are required to find, will be $= 1.936$.

Q. E. I.

Art. CLXXV. Since the firſt of the three numbers ſought is thus found to be $= 1.936$, the ſecond will be $\left(= \frac{6}{1.936}\right) = 3.099$, and the third, or greateſt, of them will be $\left(= \frac{\overline{3.099}^2}{1.936} = \frac{9.593,801}{1.936}\right) = 4.955$. And theſe numbers

numbers will anſwer the conditions of the Problem to a tolerable degree of exactneſs, their ſum 1.936 + 3.099 + 4.955 being = 9.990, which is very nearly equal to 10.

Art. CLXXVI. And, if we ſubſtitute the number 1.936 inſtead of x in the compound quantity $10x^3 - 6xx - x^4$, we ſhall find that the reſult will be pretty nearly equal to 36, or the abſolute term of the equation $10x^3 - 6xx - x^4 = 36$, by the reſolution of which the ſaid number has been obtained.

For, if x is = 1.936, we ſhall have xx (= $\overline{1.936}^2$) = 3.748,096, and x^3 (= $\overline{1.936}^3$) = 7.256,313,856, and x^4 (= $\overline{1.936}^4$) = 14.048,223,625,216, and conſequently $6xx$ (= 6 × 3.748,096) = 22.488,576, and $10x^3$ (= 10 × 7.256,313,856) = 72.563,138,560, and $10x^3 - 6xx - x^4$ (= 72.563,138,560 — 22.488,576 — 14.048,223,625,216 = 72.563,138,560 — 36.536,799,625,216) = 36.026,338,934,784; which exceeds the abſolute term 36 by only the ſmall quantity 0.026, &c. or about the 1384th part of 36. This is as great a degree of exactneſs as ought to be expected from the foregoing reſolution of the equation $10x^3 - 6xx - x^4 = 36$, becauſe the value of n, or the root of the cubick equation $n^3 + 15.75nn + 43.6875n = 440.046,875$ (upon which the ſubſequent operations of the foregoing reſolution depend,) was found only to four places of figures, or to be = 3.760.

Art. CLXXVII. We have now compleatly gone through the reſolution of the biquadratick equation $10x^3 - 6xx - x^4 = 36$ by the method of Lewis Ferrari; which has been longer and more troubleſome than the reſolution of the former equation $60x - 6xx - x^4 = 36$, becauſe of the neceſſity we were under of getting rid of the cube of the unknown quantity x, or of converting the equation $10x^3 - 6xx - x^4 = 36$ into the equation $y^4 - 31.5yy + 95y = 43.6875$, before the method of Lewis Ferrari could be applied to it: and this converſion of the firſt equation into the ſecond (which was performed in Art. 169,) required ſome tedious ſubſtitutions and reductions. And in the ſubſequent part of the reſolution there was alſo another difficulty that did not occur in the reſolution of the former equation $60x - 6xx - x^4 = 36$ by the ſame method of Lewis Ferrari; namely, the neceſſity of determining whether ay was greater or leſs than b, in order to determine whether $ay - b$, or $b - ay$, was the ſquare-root of the trinomial quantity $aayy - 2aby + bb$. So that, upon the whole, I conceive that the reader will have found the foregoing reſolution of this equation a very tedious and troubleſome piece of calculation. But by Mr. Raphſon's method of approximation we might have found the root of this equation to the ſame, or a greater, degree of exactneſs with a fourth part of the labour employed in the foregoing reſolution. This I ſhall now endeavour to make appear by exhibiting a reſolution of this ſame equation $10x^3 - 6xx - x^4 = 36$ performed by Mr. Raphſon's ſaid method.

A RESO-

A RESOLUTION OF THE BIQUADRATICK EQUATION $10x^3 - 6xx - x^4 = 36$ BY MR. RAPHSON'S METHOD OF APPROXIMATION.

Art. CLXXVIII. SINCE x is the leaſt of the three numbers ſought in the above-mentioned Arithmetical Problem; —and, by one of the conditions of the Problem, the product of the multiplication of the leaſt of theſe numbers by the ſecond is equal to 6; it follows that the leaſt number x muſt be leſs than the ſquare-root of 6, that is, than 2.449,490, &c. We will therefore, in the firſt place, ſuppoſe it to be = 2.3, and try the reſult of this ſuppoſition.

Now, if x, or the leaſt number, is = 2.3, the ſecond number will be $\left(= \frac{6}{x} = \frac{6}{2.3}\right) = 2.608$, and the third, or greateſt, number will be $\left(= \frac{\overline{2.608}^2}{2.3} = \frac{6.801,664}{2.3}\right) = 2.957$; and conſequently the ſum of the three numbers will be $(= 2.3 + 2.608 + 2.957) = 7.865$. This is much leſs than 10. Therefore the leaſt number x muſt be conſiderably leſs than 2.3, to the end that the ſecond (which is $= \frac{6}{x}$,) may be greater than before, and that the greateſt, (which is a third proportional to x and $\frac{6}{x}$,

and therefore is $= \frac{36}{x^3}$,) may likewiſe be greater than before, and that the ſum of the three may be nearly equal to 10. We will therefore, in the ſecond place, ſuppoſe x to be $= 2$, and try the reſult of this ſuppoſition.

Now, if x is $= 2$, we ſhall have $\frac{6}{x} = \left(\frac{6}{2}\right) = 3$, and $\frac{36}{x^3}\left(= \frac{36}{8} = \frac{9}{2}\right) = 4.5$; and conſequently the ſum of the three numbers x, $\frac{6}{x}$, and $\frac{36}{x^3}$ will be $(= 2 + 3 + 4.5) = 9.5$; which, as well as the former reſult, is leſs than 10, but falls ſhort of it by only 0.5. We will therefore, in the third place, ſuppoſe x to be $= 1.9$, and try the reſult of this ſuppoſition.

Now, if x is $= 1.9$, we ſhall have $\frac{6}{x}\left(= \frac{6}{1.9}\right) = 3.157$, and $\frac{36}{x^3}\left(= \frac{6}{x} \times \frac{6}{x} \times \frac{1}{x} = \frac{6}{1.9} \times \frac{6}{1.9} \times \frac{1}{1.9} = \overline{3.157}^2 \times \frac{1}{1.9} = \frac{\overline{3.157}^2}{1.9} = \frac{9.966,649}{1.9}\right) = 5.245$; and conſequently the ſum of the three numbers will be $(= 1.900 + 3.157 + 5.245) = 10.302$; which is greater than 10. Therefore 1.9 muſt be a little leſs than the true value of x. But it will be near enough to it to be made the baſis of a nearer approach to the true value of x, or the root of the equation $10x^3 - 6xx - x^4 = 36$, by Mr. Raphſon's method of approximation.

Art.

Art. CLXXIX. Let us therefore ſuppoſe x to be $= 1.9 + z$, and let us ſubſtitute $1.9 + z$ inſtead of x in the terms of the equation $10x^3 - 6xx - x^4 = 36$.

Then, ſince x is $= 1.9 + z$, we ſhall have

$xx\ (= \overline{1.9 + z}^2 = \overline{1.9}^2 + 2 \times 1.9 \times z + \&c = \overline{1.9}^2$
$+ 3.8 \times z + \&c) = 3.61 + 3.8 \times z + \&c,$

and $x^3\ (= \overline{1.9 + z}^3 = \overline{1.9}^3 + 3 \times \overline{1.9}^2 \times z + \&c$
$= \overline{1.9}^3 + 3 \times 3.61 \times z + \&c = \overline{1.9}^3 + 10.83 \times z + \&c)$
$= 6.859 + 10.83 \times z + \&c,$

and $x^4\ (= \overline{1.9 + z}^4 = \overline{1.9}^4 + 4 \times \overline{1.9}^3 \times z + \&c$
$= \overline{1.9}^4 + 4 \times 6.859 \times z + \&c$
$= \overline{1.9}^4 + 27.436 \times z + \&c)$
$= 13.0321 + 27.436 \times z + \&c.$

Therefore $10x^3$ will be $(= 10 \times \overline{6.859 + 10.83 \times z + \&c}$
$= 10 \times 6.859 + 10 \times 10.83 \times z + \&c) = 68.59 + 108.3 \times z + \&c,$

and $6xx$ will be $(= 6 \times \overline{3.61 + 3.8 \times z + \&c}$
$= 6 \times 3.61 + 6 \times 3.8 \times z + \&c)$
$= 21.66 + 22.8 \times z + \&c,$

and consequently $10x^3 - 6xx - x^4$ will be $=$

$$\left\{\begin{array}{rrl} 68.59 & + & 108.3 \times z + \&c \\ -\ 21.66 & - & 22.8 \times z - \&c \\ -\ 13.0321 & - & 27.436 \times z - \&c \end{array}\right\}$$

$$= \left\{ \begin{array}{r} 68.5900 + 108.300 \times z + \&c \\ - 34.6921 - 50.236 \times z - \&c \end{array} \right\}$$
$$= 33.8979 + 58.064 \times z + \&c.$$

But $10x^3 - 6xx - x^4$ is $= 36$.

Therefore $33.8979 + 58.064 \times z$ will alſo be $= 36$. And conſequently $58.064 \times z$ will be $(= 36 - 33.8979) = 2.1021$, and z will be $= \frac{2.1021}{58.064} = 0.036$. Therefore x, or $1.9 + z$, will be $(= 1.9 + 0.036) = 1.936$; that is, the root of the equation $10x^3 - 6xx - x^4 = 36$ will be $= 1.936$.

Q. E. I.

And thus we have found the root of the biquadratick equation $10x^3 - 6xx - x^4 = 36$, by only one proceſs of Mr. Raphſon's method of approximation, to the ſame degree of exactneſs as it was found by the former long reſolution of it by the method of Lewis Ferrari, which was attended with ſo many difficulties and ſuch a quantity of calculation. And from this and the former examples of the great uſefulneſs of this method of reſolving equations by approximation, I preſume my readers will be ready to join with me in concluding generally, that, whenever there is occaſion to reſolve a biquadratick equation of any form, whether it does, or does not, involve in it the cube of the unknown quantity, (excepting only the three equations $x^4 + qx^2 = s$, $x^4 - qx^2 = s$, and $qx^2 - x^4 = s$, which may be conſidered as quadratick equations,) and even whenever

whenever there is occaſion to reſolve a cubick equation of any kind, except the cubick equation $y^3 + qx = r$, which comes under Cardan's firſt rule, and the cubick equation $y^3 - qy = r$, in the firſt caſe of that equation, or when r is greater than $\frac{2q\sqrt{q}}{3\sqrt{3}}$, or $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, which comes under Cardan's ſecond rule, it will always be found moſt convenient and expeditious to reſolve it by this method of approximation.

Art. CLXXX. As to equations of higher orders than biquadraticks, no accurate methods of reſolving them have yet been diſcovered; and therefore in theſe equations it is abſolutely neceſſary to have recourſe to the methods of approximation to aſcertain the values of their roots. And of all theſe methods I conſider that of Mr. Raphſon as the moſt convenient. And therefore I ſhall here reprint a tract upon this ſubject which I publiſhed about three years ago at the end of a large volume in octavo (containing ſeveral uſeful mathematical tracts,) that is ſold by Mr. White, bookſeller, in Fleet Street. The title of this tract is as follows; *Obſervations on Mr. Raphſon's Method of reſolving Affected Equations of all Degrees by Approximation*; and the principal object of it is to facilitate to beginners the underſtanding of Mr. Raphſon's own treatiſe on the ſubject, and to remove ſome difficulties that may occur to them in the peruſal of it, and which ariſe chiefly from Mr. Raphſon's having, in ſome of his Problems, adopted the obſcure and perplexing doctrine of negative quantities and negative roots of equations, and made uſe of the ambiguous and myſterious language which that doctrine naturally introduces: and I

have alſo endeavoured to compare Mr. Raphſon's method of reſolving equations by approximation with that of Sir Iſaac Newton, (from which it differs but little,) and to form an eſtimate of their reſpective merits; the reſult of which compariſon is, that I conſider Mr. Raphſon's method of reſolving them as, upon the whole, more convenient than that of Sir Iſaac Newton. And I prefer it in a ſtill higher degree to the methods of reſolving affected equations by approximation recommended by Monſieur de Lagny and Dr. Halley. But of this ſubject it is unneceſſary to ſay any thing further on this occaſion, as the reader will find all I have to offer upon it ſet forth in the following Diſcourſe. And therefore I here put an end to this Appendix.

THE END.

OBSERVATIONS

ON

MR. RAPHSON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

OBSERVATIONS

ON

MR. RAPHSON'S METHOD

OF

RESOLVING AFFECTED EQUATIONS OF ALL DEGREES

BY

APPROXIMATION.

Art. I. IN the foregoing Tract I have given a pretty full explanation of Monſieur de Lagny's Method of Extracting the Roots of Numbers by Approximation, and I have likewiſe mentioned Mr. Raphſon's more ſimple and eaſy, though leſs exact, method of performing the ſame thing. But both theſe methods may be applied to the reſolution of all ſorts of equations, thoſe which are called *affected* equations *, or in which the unknown quantity

* This expreſſion of *affected equations* ſeems to require ſome further explanation. It was introduced by the celebrated Vieta, the great father and reſtorer of Algebra. He has many expreſſions peculiar to himſelf, and which have not been adopted by ſubſequent Algebraiſts. Amongſt

tity occurs in more than one term, as well as thoſe which are called *pure* equations, or in which the unknown quantity occurs in only one term, and which are reſolved by the mere extraction of the roots of given numbers. And in

Amongſt theſe are the following ones.—He calls a ſet of quantities in continual geometrical proportion, (ſuch as the quantities $1, x, x^2, x^3, x^4, x^5, x^6, x^7$, &c,) a ſet of *ſcalar* quantities, or *magnitudines ſcalares*; and, when there are ſeveral of theſe *ſcalar* quantities connected with each other by the ſigns + and —, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$,) he calls the higheſt quantity, or that which is fartheſt in the ſcale of quantities $1, x, x^2, x^3, x^4, x^5, x^6, x^7$, &c, (to wit, the quantity x^5 in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$,) *the power* of the fundamental quantity x, or of the ſecond term in the ſaid ſcale; and he calls the lower ſcalar quantities, which are involved in the ſecond and third terms of the ſaid compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities x^4 and x^3, (or, in our preſent language, the inferiour powers of x,) ſcalar quantities of a *parodic* degree to x^5, or *the power* of the fundamental quantity x. This word *parodic* I take to be derived (though Vieta does not tell us ſo,) from the Greek words παρὰ and ὁδὸς, which ſignify *near* and *a way*, or *road*, becauſe theſe inferiour ſcalar quantities, x^3 and x^4, lie *in the way* as you paſs along in the ſcale of the aforeſaid quantities $1, x, x^2, x^3, x^4, x^5, x^6, x^7$, &c, from 1 to x^5, which he calls *the power* of x in the ſaid compound quantity $x^5 + ax^4 - b^2x^3$. Theſe inferiour ſcalar quantities x^3 and x^4 are therefore *parodic*, or *ſituated in the way to*, or are *leading to*, the ſaid *power*, or higher ſcalar quantity, x^5. He then proceeds to define *a pure power* and *an affected power*, and tells us, that *a pure power* is a ſcalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* ſcalar quantity, and that *an affected power* is a ſcalar quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, ſcalar quantities, combined with co-efficients that raiſe them to the ſame dimenſion as the *power* itſelf, or make them *homogeneous* to it, and conſequently

in all affected equations beyond biquadraticks, or thoſe of the fourth power, theſe methods of approximation are the only methods that can be taken for diſcovering their roots, or the values of the unknown quantities contained in them. And even in cubick and biquadratick equations, though

quently capable of being added to it, or ſubtracted from it. Thus x^5 alone is *a pure power* of x, namely, its fifth power; and $x^5 + ax^4 - b^2x^3$ is *an affected power* of x, namely, its fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour* ſcalar quantities, x^3 and x^4, which are multiplied into bb and a, in order to make them *homogeneous to*, or *of the ſame dimenſion with*, x^5 itſelf, and conſequently capable of being added to it, or ſubtracted from it. And he ſeems to have uſed the word *affected* in ſpeaking of ſuch a compound quantity as $x^5 + ax^4 - b^2x^3$, becauſe the magnitude of the higheſt power of x, to wit, x^5, was changed, or altered, or made greater, or leſs, than it would otherwiſe be, by the addition of the parodic quantity ax^4, and the ſubtraction of the parodic quantity b^2x^3; which increaſe, or diminution, or change, in the principal quantity x^5 he ſeemed to think might be well expreſſed by ſaying it was *affected* by its connection with the ſaid parodic quantities. There are ſome expreſſions in his book that ſatisfy me that this was the idea he annexed to the word *affected*. See Schooten's edition of Vieta's Works, publiſhed at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expreſſions *a pure power* and *an affected power*, the meaning of the correſponding expreſſions of *a pure equation* and *an affected equation* follows from it of courſe: *a pure equation* ſignifying an equation in which a pure power of an unknown quantity is declared to be equal to ſome known quantity; ſuch as the equation $x^5 = 79$; and *an affected equation* ſignifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, ſome inferiour powers of the ſame unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the ſaid higheſt power of the ſaid unknown quantity,) is declared to be equal to ſome known quantity; ſuch

though particular methods have been invented by mathematicians, for the accurate reſolution of moſt of the caſes of theſe equations, (to wit, the rules called Cardan's Rules for the reſolution of moſt caſes of cubick equations, and the rules invented by Lewis Ferrari of Bologna in Italy, about the year 1545, and explained at large in Bombelli's Algebra, in the year 1579, and thoſe afterwards invented by M. Des Cartes, and publiſhed in his Geometry in the year 1637, for the reſolution of biquadratick equations, by the mediation of cubick equations,) it will be found that theſe methods of approximation will, for the moſt part, enable us to find the values of their roots to any

ſuch as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expreſſion *an affected equation*. But, as the language of Vieta has not been adopted by ſubſequent writers of Algebra, I ſhould think it would be more convenient to call them by ſome other name. And, perhaps, thoſe of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, becauſe they would be equations in which a *binomial* quantity, or quantity conſiſting of two terms that involved the unknown quantity x, is declared to be equal to a known quantity; and, for a like reaſon, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^3x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I obſerve, in his excellent new Mathematical and Philoſophical Dictionary, juſt now publiſhed, (Feb. 2, 1795,) calls them *compound* equations; which is likewiſe a very proper name for them, and leſs obſcure than that of *affected* equations. And Mr. Kerſey, in his excellent Treatiſe of Algebra, publiſhed in the year 1673, likewiſe calls them *compound* equations.

propoſed degree of exactneſs, with leſs trouble than the particular and accurate methods above-mentioned, which have been invented for that purpoſe. So that theſe methods of reſolving equations by approximation ought to be conſidered as of the higheſt utility, and as being abſolutely neceſſary to the completion of the Doctrine of the Reſolution of Algebräick Equations, which is the moſt important branch of the Science of Algebra.

Art. II. But it is not ſo eaſy to determine, which of theſe two methods of approximation, Mr. Raphſon's, or Mr. de Lagny's, deſerves to be preferred to the other on theſe occaſions. Mr. Raphſon's is certainly much ſimpler than the other, becauſe it proceeds by conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as being only a ſimple equation, and reſolving it accordingly, or by the mere operation of Diviſion; whereas, in Mr. de Lagny's method, the ſaid new, or transformed, equation is conſidered as a quadratick equation, and reſolved accordingly; which, when a (or the firſt near value of the root, that is ſuppoſed to be already known,) is a number conſiſting of five, or ſix, figures, produces a great deal of labour, and often a great deal of perplexity. I am therefore inclined to give the preference to Mr. Raphſon's method in reſolving all affected equations, more eſpecially when the number a conſiſts of more than two figures: but it muſt be confeſſed that the celebrated Dr. Halley (who had much experience, and was an excellent judge of theſe matters,) was of a different opinion, and gave the preference to Mr. de Lagny's method, which

which he has therefore taken the pains to explain in a better manner than had been done by Mr. de Lagny himſelf, and likewiſe to illuſtrate by examples, in his Tract in the Philoſophical Tranſactions, Number 210, intitled, "*A New, Exact, and Eaſy, Method of finding the "Roots of any Equations Generally, and that without any "previous Reduction,*" which was publiſhed in the year 1694. On the other hand we may obſerve, that Mr. Raphſon always continued to give his own method the preference, after the publication of the tracts of Mr. de Lagny and Dr. Halley upon the ſubject, as well as before their publication, when he tells us he had himſelf had the thought of adopting the principle which was afterwards followed by Mr. de Lagny and Dr. Halley, of treating the transformed equation as a quadratick equation, but had deliberately rejected it on account of the greater eaſe and ſimplicity of the other method; in which the ſaid transformed equation is conſidered and treated as a ſimple equation. And Sir Iſaac Newton, in his method of reſolving equations by approximation, (which differs very little from Mr. Raphſon's,) ſeems alſo to prefer Mr. Raphſon's practice, of treating the transformed equation as a mere ſimple equation, to that of Mr. de Lagny and Dr. Halley, of treating the ſaid equation as a quadratick equation. I therefore cannot but recommend it to all young Algebräiſts to ſtudy Mr. Raphſon's excellent Treatiſe on this ſubject, intitled, *Analyſis Æquationum Univerſalis*, with great attention, and to endeavour to make themſelves maſters of it, by going carefully through all the examples given in it, and performing all the arithmetical operations contained in them. And I will venture to ſay that they will

will thereby acquire more uſeful knowledge in Algebra, towards the buſineſs of reſolving affected, or compound, or multinomial, equations, than by reading all that has been written by Harriot and Des Cartes, and his learned Commentator Van Schooten, and all his other Commentators, and their numerous followers, on the boaſted doctrine of the Generation of Equations one from another, by ſuppoſing $x - a$ to be $= 0$, and $x - b$ to be $= 0$, and $x - c$ to be $= 0$, and $x + d$ to be $= 0$, and $x + e$ to be $= 0$, and ſo on; and then multiplying the binomial quantities $x - a$, $x - b$, $x - c$, $x + d$, $x + e$, &c, into each other, and likewiſe all the abſtruſe and intricate matter that has been delivered by Sir Iſaac Newton, and Mr. Graveſende and Mr. Mac Laurin, and other learned Algebräiſts of modern times, on the invention of Diviſors, which is grounded on that doctrine of the Generation of Equations from each other.

Art. III. Yet in reading this excellent Treatiſe of Mr. Raphſon, which I ſo much recommend, there will now and then occur ſome difficulties which are not inherent in the ſubject itſelf, but which might have been avoided, if Mr. Raphſon had not unfortunately adopted the perplexing doctrines of modern writers of Algebra, about negative quantities and negative roots of equations. The quantities called *negative* are ſuch as it is impoſſible to form any clear idea of, being defined, by Sir Iſaac Newton and other Algebräiſts *, to be ſuch quantities as are *leſs than nothing*,

* Quantitates vel *Affirmativæ* ſunt, ſeu majores Nihilo, vel *Negativæ*, ſeu Nihilo minores.—Newton's *Arithmetica Univerſalis*, page 3.

nothing, or as *arise from the subtraction of a greater quantity from a lesser*, which is an operation evidently impossible to be performed: and, as to the *negative roots* of an equation, they are in truth the real and positive roots of another equation consisting of the same terms as the first equation, but with different signs + and — prefixed to some of them; so that, when writers of Algebra talk of the negative roots of an equation, they, in fact, jumble two different equations together, and suppose the proposed, or first, equation to have not only its own proper roots (which they call its *affirmative*, or *positive*, roots,) but to have likewise the roots of a different equation, which they call its *negative* roots. Thus, for example, they would say, that the quadratick equation $xx + 4x = 320$, has two roots, to wit, the positive, or affirmative, root, + 16, and the negative root, — 20. But this latter number, 20, is, in truth, the root of a different equation, to wit, of the equation $xx - 4x = 320$. So that this kind of absurd and fantastick language only tends to the confounding together the two different equations $xx + 4x = 320$, and $xx - 4x = 320$, and considering them as if they were one and the same equation. Now this perplexing language is unfortunately used by Mr. Raphson in this valuable Treatise, and tends to throw an air of mystery and

When a greater quantity is taken from a lesser of the same kind, the remainder becomes of the opposite kind.——*Mac Laurin's Algebra*, page 5.

An affirmative quantity is a quantity greater than nothing, and is known by this sign, +; a negative quantity is a quantity less than nothing, and is known by this sign, —.——*Saunderson's Algebra*, Vol. I. page 50, Article 2.

obscurity

obſcurity upon ſome of the Problems ſolved in it, from which they would otherwiſe have been intirely free. As a proof of the truth of this obſervation, I ſhall here inſert one of the ſaid Problems, the ſolution of which is by this means rendered ſo obſcure, that I had a good deal of trouble to find out the meaning of it; though, if this language had been avoided, and the proper and natural language, belonging to the conditions of the Problem, had been uſed in its ſtead, there could not have been the leaſt difficulty in underſtanding it. This Problem is the 24th, in page 32 of the 2d edition of the book, and is, *verbatim et litteratim*, as follows.

PROBLEMA XXIV.

ÆQUATIONUM QUINTÆ POTESTATIS ADFECTARUM SOLUTIO.

Proponatur $-aaaaa + 7aaaa - 20aaa + 155aa = 10{,}000$.

Hoc eſt, $-aaaaa + baaaa - caaa + daa = f$.

Theor. $x = \dfrac{f + ggggg + cggg - bgggg - dgg}{4bggg + 2dg - 5gggg - 3cgg}$

Sit $g = -5$

$f + ggggg + cggg - bgggg - dgg = -3875$

$4bggg + 2dg - 5gggg - 3cgg = -9675$) $(-3875{,}0)$ $+{,}4 = x$

$-5{,}$

$+{,}4$

$g = -4{,}6$

$f + ggggg + cggg - bgggg - dgg = -420,36896$
$4bggg + 2dg - 5gggg - 3cgg = -7659,736) -420,36896$
$(+,055 = x$

$-4,6$
$+,055$

$g = -4,545$

$f + ggggg + cggg - bgggg - dgg = -5,9603594654625625$
$4bggg + 2dg - 5gggg - 3cgg = -7410,748) -5,9603594$
$(+,0008c428 = x$

$-4,545$
$+,000,804,28$

$a = -4,544,195,72$

To this ſolution I have, in my copy of Mr. Raphſon's Tract, ſubjoined the following Note :

Numerus 4.544,195,72 eſt radix æquationis $a^3 + 7a^4 + 20a^3 + 155a^2 = 10,000$; quod hic obſcurè innuitur ſub ſpecie radicis negativæ æquationis $-a^5 + 7a^4 - 20a^3 + 155a^2 = 10,000$. Omnes ferè difficultates quibus permulti cultioris ingenii viri ab Algebrâ diſcendâ et excolendâ deterrentur, ex hiſce radicibus negativis et aliis quantitatibus negativis, ſeu (ut hodierni Algebræ ſcriptores abſurdè loquuntur,) nihilo minoribus, ortum habent.

In this Problem the letter a is uſed for the unknown quantity, or root of the equation, which is uſually denoted by the letter x; and the letter g is uſed for the firſt near value

value of the root of the equation, which in the two foregoing Tracts has been denoted by the letter a; and the letter x is used for the difference between g, the first near value of the root of the equation, and z, its true value, which difference has been denoted in the two foregoing Tracts by the letter z. So that, if we express the enunciation of the foregoing Problem in the notation that has been used in the two foregoing Tracts, it will be as follows:

Proponatur $-xxxxx+7xxxx-20xxx+155xx=10{,}000$,

Sive $-x^5+7x^4-20x^3+155xx=10{,}000$,

Hoc est, $-xxxxx+bxxxx-cxxx+dxx=f$,

Sive $-x^5+bx^4-cx^3+dx^2=f$.

Theor. $z=\dfrac{f+aaaaa+caaa-baaaa-daa}{4baaa+2da-5aaaa-3caa}$,

or $z=\dfrac{f+a^5+ca^3-ba^4-da^2}{4ba^3+2da-5a^4-3ca^2}$.

Art. IV. Here, then, the equation proposed by Mr. Raphson to be resolved, is said to be $-x^5+7x^4-20x^3+155xx=10{,}000$, or $155xx-20x^3+7x^4-x^5=10{,}000$. But this is not the equation he resolves; and, indeed, it is not a possible equation, because the greatest possible magnitude of the compound quantity $155xx-20x^3+7x^4-x^5$ is that which it has when the infinitely small increment of the binomial quantity $20x^3+x^5$ becomes equal to the contemporary increment of the binomial quantity $155xx+7x^4$, that is, (if we put $\dot{x}$, or x with a point placed over it, for the infinitely small incre-

ment of x,) when $20 \times 3x^2\dot{x} + 5x^4\dot{x}$ becomes equal to $155 \times 2x\dot{x} + 7 \times 4x^3\dot{x}$, or when $60x^2 + 5x^4$ is $= 310x + 28x^3$, or when $60x + 5x^3$ is $= 310 + 28xx$, or when $5x^3 - 28xx + 60x$ is $= 310$, or when $x^3 - \frac{28xx}{5} + 12x$ is $= 62$, or when $x^3 - 5.6xx + 12x$ is $= 62$; and that is when x is nearly $= 5.5$; at which time the compound quantity $155xx - 20x^3 + 7x^4 - x^5$ will be nearly equal to 2733, as will appear by ſubſtituting 5.5 inſtead of x in the terms of the ſaid quantity $155xx - 20x^3 + 7x^4 - x^5$: and this quantity 2733 (which is the greateſt poſſible magnitude of the compound quantity $155xx - 20x^3 + 7x^4 - x^5$,) is very much leſs than 10,000, or the abſolute term of the equation $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$, and conſequently the ſaid equation is impoſſible. But Mr. Raphſon, though he ſets down this equation $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$, as the equation that is to be reſolved, yet really means to reſolve a quite different equation, to wit, the equation that reſults from ſuppoſing x to be a negative quantity, or from ſubſtituting the powers of $-x$, to wit, $+xx$, $-x^3$, $+x^4$, and $-x^5$, in the terms of the ſaid equation $155xx - 20x^3 + 7x^4 - x^5 = 10{,}000$, inſtead of the like powers of $+x$, to wit, $+xx$, $+x^3$, $+x^4$, and $+x^5$; by which ſubſtitution the ſaid equation will be converted into the equation $155 \times + xx - 20 \times - x^3 + 7 \times + x^4 - 1 \times - x^5 = 10{,}000$, or $155xx + 20x^3 + 7x^4 + x^5 = 10{,}000$, which is evidently a poſſible equation, and of which the root is 4.544,195,72, or the ſame number which he obtains by his ſolution of the Problem, and which, with the ſign $-$ prefixed to it, he calls the negative root of the propoſed equation $155xx -$

$20x^3$

$20x^3 + 7x^4 - x^5 = 10{,}000$. Now all this perplexity would have been avoided, if Mr. Raphſon had propoſed at firſt to find the root, or, in the language of modern writers of Algebra, the *affirmative*, or *poſitive*, root, of the equation $155xx + 20x^3 + 7x^4 + x^5 = 10{,}000$, or $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which equation is evidently poſſible, and can have only one root. And then all the ſteps of his ſolution would have been clear and eaſy, as will appear by reſolving this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$ according to the principles of his method; which may be done in the manner following:

THE RESOLUTION OF THE AFFECTED EQUATION $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, BY MR. RAPHSON'S METHOD OF APPROXIMATION.

Art. V. IN conſidering this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, it is, in the 1ſt place, eaſy to ſee that x muſt be greater than 1. For, if we ſuppoſe x to be $= 1$, we ſhall have $xx = 1$, and $x^3 = 1$, and $x^4 = 1$, and $x^5 = 1$; and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ will be $= 1 + 7 + 20 + 155 = 183$; which is very much leſs than the abſolute term 10,000. Therefore 1 muſt be much leſs than x.

In the ſecond place, if we ſuppoſe x to be $= 10$, we ſhall have $xx = 100$, and $x^3 = 1000$, and $x^4 = 10{,}000$,

and $x^5 = 100{,}000$; ſo that x^4 alone will be equal to the abſolute term 10,000, and conſequently $x^5 + 7x^4 + 20x^3 + 155x$ muſt be very much greater than the ſaid abſolute term; and conſequently 10 muſt be much greater than x.

Thirdly, ſince x is leſs than 10 and greater than 1, let us ſuppoſe it to be equal to 5. Then we ſhall have $xx = 25$, and $x^3 = 125$, and $x^4 = 625$, and $x^5 = 3125$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 3125 + 7 \times 625 + 20 \times 125 + 155 \times 25 = 3125 + 4375 + 2500 + 3875$) $= 13{,}875$; which is greater than the abſolute term 10,000. Therefore 5 is greater than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

We will therefore, in the 4th place, ſuppoſe x to be $= 4$. And then we ſhall have $xx = 16$, and $x^3 = 64$, and $x^4 = 256$, and $x^5 = 1024$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ ($= 1024 + 7 \times 256 + 20 \times 64 + 155 \times 16 = 1024 + 1792 + 1280 + 2480$) $= 6576$; which is leſs than the abſolute term 10,000. Therefore 4 is leſs than the true value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

It appears therefore that the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$ is greater than 4, but leſs than 5. And either of theſe values might very well ſerve for a firſt near value of the ſaid root, or for the baſis of a further approximation to it. Mr. Raphſon makes choice of 5, which is greater than the truth.

Art.

Art. VI. Let us then ſuppoſe a, or the firſt near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, to be $= 5$; and let z be the difference by which it exceeds the true value of x. Then will x be $= a - z$, and conſequently xx will be $(= a - z|^2) = aa - 2az +$ &c, and x^3 will be $(= a - z|^3) = a^3 - 3a^2z +$ &c, and x^4 will be $(= a - z|^4) = a^4 - 4a^3z +$ &c, and x^5 will be $(= a - z|^5) = a^5 - 5a^4z +$ &c. Therefore $x^5 + 7x^4 + 20x^3 + 155xx$ will be $=$

$$\left\{\begin{array}{rr} & a^5 - 5a^4z + \&c, \\ + & 7 \times a^4 - 4a^3z + \&c, \\ + & 20 \times a^3 - 3a^2z + \&c, \\ + & 155 \times aa - 2az + \&c, \end{array}\right\}$$

$$= \left\{\begin{array}{rr} & a^5 - 5a^4z + \&c, \\ + & 7a^4 - 28a^3z + \&c, \\ + & 20a^3 - 60a^2z + \&c, \\ + & 155aa - 310az + \&c. \end{array}\right\} \ldots$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z - 28a^3z - 60a^2z - 310az +$ &c, will alſo be $= 10{,}000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) we ſhall have $a^5 + 7a^4 + 20a^3 + 155aa = 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$, or (becauſe a is $= 5$, and conſequently $a^5 + 7a^4 + 20a^3 + 155aa$ is $= 13{,}875$, as has been ſhewn in Art. 5,) we ſhall have $13{,}875 = 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$, and conſequently (ſubtracting 10,000 from both ſides,) $3875 = 5a^4z + 28a^3z + 60a^2z + 310az = z \times 5a^4 + 28a^3 + 60a^2 + 310a$. Therefore

fore z will be $= \frac{3875}{5a^4 + 28a^3 + 60a^2 + 310a}$ $(= \frac{3875}{5 \times \overbrace{5}^{4} + 28 \times \overbrace{5}^{3} + 60 \times \overbrace{5}^{2} + 310 \times 5} =$ $\frac{3875}{5 \times 625 + 28 \times 125 + 60 \times 25 + 310 \times 5} =$ $\frac{3875}{3125 + 3500 + 1500 + 1550} = \frac{3875}{9675}) = 0.4$. Therefore $a - z$, or x, will be $(= a - 0.4 = 5.0 - 0.4) = 4.6$; and 4.6 will be a second near value of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

We muſt next try whether this ſecond near value of x is greater or leſs than its true value; and for this purpoſe we muſt ſubſtitute it, inſtead of x, in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$.

Now, if we ſuppoſe x to be $= 4.6$, we ſhall have xx $(= \overbrace{4.6}^{2}) = 21.16$, and x^3 $(= \overbrace{4.6}^{3}) = 97.336$, and x^4 $(= \overbrace{4.6}^{4}) = 447.7456$, and x^5 $(= \overbrace{4.6}^{5}) = 2059.62976$, and $155xx$ $(= 155 \times 21.16) = 3279.80$, and $20x^3$ $(= 20 \times 97.336) = 1946.720$, and $7x^4$ $(= 7 \times 447.7456) = 3134.2192$, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx$ $(= 2059.62976 + 3134.2192 + 1946.720 + 3279.80) = 10{,}420.36896$; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Therefore 4.6 will be greater than the true value of x in that equation.

Art.

Art. VII. To find a third near value of the root of this equation, let a be ſuppoſed to be $= 4.6$, and z be the difference by which a, or 4.6, exceeds the true value of the ſaid root.

Then we ſhall have, as before, $x = a - z$, and conſequently $xx\ (= \overline{a - z}^2) = aa - 2az +$ &c, and $x^3\ (= \overline{a - z}^3) = a^3 - 3a^2z +$ &c, and $x^4\ (= \overline{a - z}^4) = a^4 - 4a^3z +$ &c, and $x^5\ (= \overline{a - z}^5) = a^5 - 5a^4z +$ &c, and $7x^4\ (= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z +$ &c, and $20x^3\ (= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z +$ &c, and $155xx\ (= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az +$ &c, and $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrl} & a^5 - & 5a^4z + \&c, \\ + & 7a^4 - & 28a^3z + \&c, \\ + & 20a^3 - & 60a^2z + \&c, \\ + & 155aa - & 310az + \&c. \end{array}\right\}$$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $\left\{\begin{array}{rrl} & a^5 - & 5a^4z + \&c, \\ + & 7a^4 - & 28a^3z + \&c, \\ + & 20a^3 - & 60a^2z + \&c, \\ + & 155aa - & 310az + \&c, \end{array}\right\}$ will likewiſe be $= 10{,}000$, and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been ſhewn in the laſt article, that $a^5 + 7a^4 + 20a^3$

$+ 20a^3 + 155aa$, or $4.6^5 + 7 \times 4.6^4 + 20 \times 4.6^3 + 155 \times 4.6^2$, is $= 10{,}420.36896$.

Therefore 10,420.36896 will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and consequently (subtracting 10,000 from both sides of the equation,) 420.36896 will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ $(= 5 \times 4.6^4 \times z + 28 \times 4.6^3 \times z + 60 \times 4.6^2 \times z + 310 \times 4.6 \times z = 5 \times 447.7456 \times z + 28 \times 97.336 \times z + 60 \times 21.16 \times z + 310 \times 4.6 \times z = 2238.7280 \times z + 2725.408 \times z + 1269.60 \times z + 1426.0 \times z) = 7659.7360 \times z$, and consequently z will be $\left(= \frac{420.36896}{7659.7360}\right)$ $= 0.0548$, or nearly 0.055. Therefore x, or $a - z$, or $4.6 - z$, will be nearly $(= 4.6 - 0.055,) = 4.545$; and consequently this number 4.545 will be a third near value of the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

Now let this number 4.545 be ſubſtituted inſtead of xx in the compound quantity $x^5 + 7x^4 + 20x^3 + 155xx$, in order to diſcover whether the reſult will be greater, or leſs, than 10,000, or the abſolute term of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Now, if x be ſuppoſed to be $= 4.545$, we ſhall have xx $(= 4.545^2) = 20.657{,}025$, and x^3 $(= 4.545^3) = 93.886{,}178{,}625$, and x^4 $(= 4.545^4) = 426.712{,}681{,}850{,}625$, and x^5 $(= 4.545^5) = 1939.409{,}139{,}011{,}090{,}625$, and conſequently

quently $7x^4$ (= 7 × 426.712,681,850,625) = 2986.988,772,954,375, and $20x^3$ (= 20 × 93.886,178,625) = 1877.723,572,500, and $155xx$ (= 155 × 20.657,025) = 3201.838,875, and $x^5 + 7x^4 + 20x^3 + 155xx$ (= 1939.409,139,011,090,625 + 2986.988,772,954,375 + 1877.723,572,500 + 3201.838,875) = 10,005.960,359,465,465,625; which is greater than 10,000, or the abſolute term of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$. Therefore 4.545 will be greater than the true value of x in that equation.

Art. VIII. To find a fourth near value of the root of this equation $x^5 + 7x^4 + 20x^3 + 155xx = 10,000$, let a be ſuppoſed to be = 4.545, and z be ſuppoſed to be the difference by which a, or 4.545, exceeds the true value of the ſaid root.

Then we ſhall, as before, have $x = a - z$, and conſequently xx $(= \overline{a - z}^2) = aa - 2az$ + &c, and x^3 $(= \overline{a - z}^3) = a^3 - 3a^2z$ + &c, and x^4 $(= \overline{a - z}^4) = a^4 - 4a^3z$ + &c, and x^5 $(= \overline{a - z}^5) = a^5 - 5a^4z$ + &c, and $7x^4$ $(= 7 \times \overline{a^4 - 4a^3z + \&c,}) = 7a^4 - 28a^3z$ + &c, and $20x^3$ $(= 20 \times \overline{a^3 - 3a^2z + \&c,}) = 20a^3 - 60a^2z$ + &c, and $155xx$ $(= 155 \times \overline{aa - 2az + \&c,}) = 155aa - 310az$ + &c, and conſequently $x^5 + 7x^4 + 20x^3 + 155xx =$

$$\left\{\begin{array}{rrrr} & a^5 & - 5a^4z & + \&c, \\ + & 7a^4 & - 28a^3z & + \&c, \\ + & 20a^3 & - 60a^2z & + \&c, \\ + & 155a^2 & - 310az & + \&c. \end{array}\right\}$$

But

$+ 20a^3 + 155aa$, or $\overline{4.6}^5 + 7 \times 4.6^4 + 155 \times \overline{4.6}^3$, is = 10,426.36896.

Therefore 10,420.36896 will be = $28a^3z + 60a^2z + 310az$; and conf 10,000 from both ſides of the equa be = $5a^4z + 28a^3z + 60a^2z +$ $z + 28 \times \overline{4.6}^3 \times z + 60 \times$ $\times z = 5 \times 447.7456 \times z +$ $\times 21.16 \times z + 310 \times 4.6 \times$ 2725.408 × z + 1269.60 ×

7360 × z, and conſequen

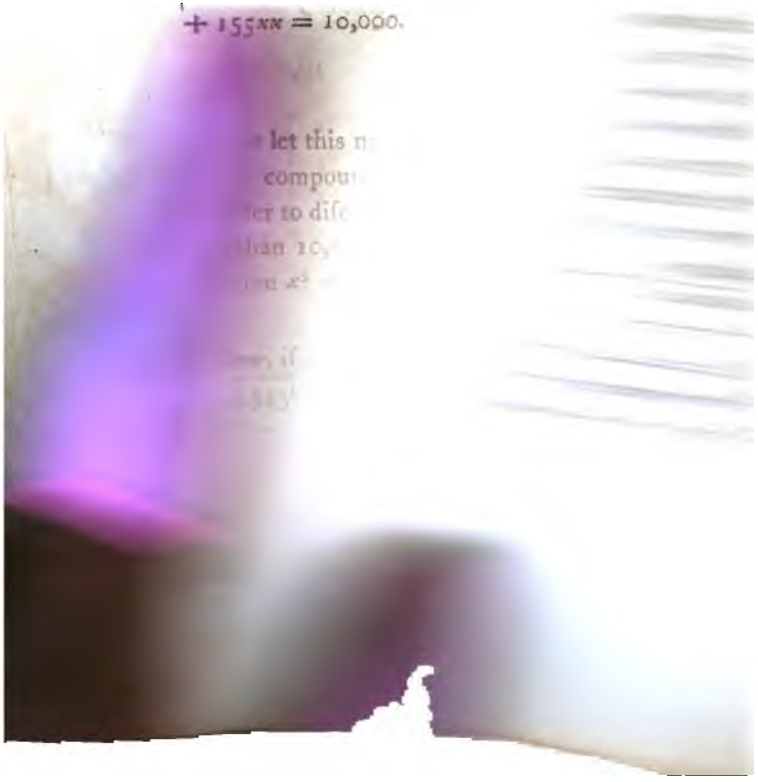
= 0.0548, or nearly 0.0

4.6 — z, will be nearly (

conſequently this num

of the root of the p

$+ 155nn = 10,000.$



z, agrees with the number
its figures.

ng reſolution of the equation
$x = 10{,}000$, has been performed
to ſet forth, in as clear a manner
ſonings upon which the arithme-
it are grounded, as well as the ſaid
And by ſo doing the ſubject is
er than in Mr. Raphſon's very con-
way of treating it, (in which all the
ped, and only the arithmetical opera-
) that, though the above reſolution
on is three, or four, times as long as
I am fully perſuaded that it may be
ood in a third, or fourth, part of the time
to a thorough comprehenſion of Mr.
lution of it; even if he had not puzzled
lking of the negative root of the equation
$20x^3 + 155xx = 10{,}000$. But that this
more clearly, I will now repeat the fore-
n of this equation in the ſtyle and manner
ſon, by omitting the ſeveral reaſonings ſet
oregoing articles, and making uſe of a Canon,
, for the purpoſe of computing the ſecond,
ourth values of z, in the ſame manner as Mr.
as done.

X. Since each of the three firſt ſucceſſive near
of x, or the root of the propoſed equation $x^5 + 7x^4$
$+ 20x^3$

But $x^5 + 7x^4 + 20x^3 + 155xx$ is $= 10{,}000$.

Therefore $a^5 + 7a^4 + 20a^3 + 155aa - 5a^4z +$ &c, $- 28a^3z +$ &c, $- 60a^2z +$ &c, $- 310az +$ &c, will likewiſe be $= 10{,}000$; and conſequently (adding $5a^4z + 28a^3z + 60a^2z + 310az$ to both ſides,) $a^5 + 7a^4 + 20a^3 + 155aa$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$.

But it has been ſhewn in the laſt article, that $a^5 + 7a^4 + 20a^3 + 155aa$, or $\overline{4.545}^5 + 7 \times \overline{4.545}^4 + 20 \times \overline{4.545}^3 + 155 \times \overline{4.545}^2$, is $= 10{,}005.960{,}359{,}465{,}465{,}625$.

Therefore $10{,}005.960{,}359{,}465{,}465{,}625$ will be $= 10{,}000 + 5a^4z + 28a^3z + 60a^2z + 310az$; and conſequently (ſubtracting 10,000 from both ſides,) $5.960{,}359{,}465{,}465{,}625$ will be $= 5a^4z + 28a^3z + 60a^2z + 310az$ $(= 5 \times \overline{4.545}^4 \times z + 28 \times \overline{4.545}^3 \times z + 60 \times \overline{4.545}^2 \times z + 310 \times 4.545 \times z = 5 \times 426.712{,}681{,}850{,}625 \times z + 28 \times 93.886{,}178{,}625 \times z + 60 \times 20.657{,}025 \times z + 310 \times 4.545 \times z = 2133.563{,}409{,}253{,}125 \times z + 2628.813{,}001{,}500 \times z + 1239.421{,}500 \times z + 1408.950 \times z) = 7410.747{,}910{,}753{,}125 \times z$. Therefore z will be $(= \frac{5.960{,}359{,}465{,}465{,}625}{7410.747{,}910{,}753{,}125}) = 0.000{,}804{,}28$, and x, or $a - z$, or $4.545 - z$, will be $(= 4.545{,}000{,}00 - 0.000{,}804{,}28) = 4.544{,}195{,}72$. Therefore $4.544{,}195{,}72$ will be a fourth near value of the root of the propoſed equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Q. E. I.

This

This number 4.544,195,72, agrees with the number found by Mr. Raphſon, in all its figures.

Art. IX. The foregoing reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, has been performed at great length, in order to ſet forth, in as clear a manner as poſſible, the ſeveral reaſonings upon which the arithmetical operations uſed in it are grounded, as well as the ſaid operations themſelves. And by ſo doing the ſubject is rendered ſo much eaſier than in Mr. Raphſon's very conciſe and compreſſed way of treating it, (in which all the reaſonings are dropped, and only the arithmetical operations are exhibited,) that, though the above reſolution of the ſaid equation is three, or four, times as long as Mr. Raphſon's, yet I am fully perſuaded that it may be read and underſtood in a third, or fourth, part of the time that is neceſſary to a thorough comprehenſion of Mr. Raphſon's reſolution of it; even if he had not puzzled the matter by talking of the negative root of the equation $-x^5 + 7x^4 - 20x^3 + 155xx = 10{,}000$. But that this may appear the more clearly, I will now repeat the foregoing reſolution of this equation in the ſtyle and manner of Mr. Raphſon, by omitting the ſeveral reaſonings ſet forth in the foregoing articles, and making uſe of a Canon, or Theorem, for the purpoſe of computing the ſecond, third, and fourth values of z, in the ſame manner as Mr. Raphſon has done.

Art. X. Since each of the three firſt ſucceſſive near values of x, or the root of the propoſed equation $x^5 + 7x^4 + 20x^3$

$+ 20x^3 + 155xx = 10{,}000$, from which the next near values of it are derived, to wit, the three numbers 5, 4.6, and 4.545, and which are ſucceſſively denoted by the letter a, is greater than the true value of x in the ſaid equation, or than the root of the ſaid equation, it follows that the ſecond, and third, and fourth near values of x will, each of them, be ſucceſſively denoted by the reſidual quantity $a - z$; and conſequently, by applying the reaſonings uſed in Art. 6, in order to obtain the values of z, and of $a - z$, or x, we ſhall find that z will be, ſucceſſively, nearly equal to the value of the fraction $\frac{a^5+7a^4+20a^3+155aa-10{,}000}{5a^4+28a^3+60a^2+310a}$, and, therefore, that $a - z$, or x, will be, ſucceſſively, nearly equal to the value of the quantity a — the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155a^2 - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. This, then, is the Theorem, or Canon, by the application of which we are to compute the ſecond, and third, and fourth, near values of $a - z$, or x, after taking 5 for the firſt near value of it, or for the firſt value of a.

Now, if a is $= 5$, we ſhall have $z =$ the fraction $\frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{3875}{9675} = 0.4$. Therefore $a - z$ will be $(= 5 - 0.4) = 4.6$; which will therefore be the ſecond near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Secondly, if a be $= 4.6$, we ſhall have $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a} = \frac{420.36896}{7659.7360}$

$= 0.0548$, or, nearly, 0.055. Therefore $a - z$ will be $(= 4.6 - 0.055) = 4.545$; which will therefore be the third near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$.

Thirdly, if a be $= 4.545$, we ſhall have $z = \frac{a^5+7a^4+20a^3+155aa-10{,}000}{5a^4+28a^3+60a^2+310a} = \frac{5.960{,}359{,}465{,}465{,}625}{7410.747{,}910{,}753{,}125}$ $= 0.000{,}804{,}28$. Therefore $a - z$ will be $(= 4.545 - 0.000{,}804{,}28) = 4.544{,}195{,}72$; which will therefore be the fourth near value of x, or of the root of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. Q. E. I.

Art. XI. Mr. Raphſon's Canon, or Theorem, for the value of z, is expreſſed more conciſely than the foregoing Theorem, $z = \frac{a^5 + 7a^4 + 20a^3 + 155aa - 10{,}000}{5a^4 + 28a^3 + 60a^2 + 310a}$. For he uſes the letters b, c, d, and f, for the co-efficients 7, 20, and 155, of the fourth, third, and ſecond, power of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, and for 10,000, the abſolute term of that equation, reſpectively; which produces the following Canon, or Theorem, for the value of z, to wit, $z = \frac{a^5+ba^4+ca^3+da^2-f}{5a^4+4ba^3+3ca^2+2da}$. But it appears to me that, though we may ſeem to gain ſomething in point of brevity by uſing this very general notation, we loſe as much in the article of perſpicuity, which is a matter of much greater importance. However, this latter reſolution of the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$, which is expreſſed in Mr. Raphſon's conciſe ſtyle and manner, and the foregoing

more explicit reſolution of it in Art. 5, 6, 7, and 8, (in which the reaſonings, on which the ſeveral arithmetical operations are grounded, are diſtinctly ſet forth and repeated,) are, both of them, the ſame in ſubſtance, and are, as I believe, the very beſt method that can be taken for diſcovering the root of the ſaid equation.

Art. XII. It has been obſerved above in Art. 2, that Sir Iſaac Newton's method of reſolving numeral equations by approximation differed but little from Mr. Raphſon's, both methods being founded on the ſame principle of conſidering the new, or transformed, equation, (reſulting from the ſubſtitution of $a + z$, or $a - z$, inſtead of x, in the original equation,) as a mere ſimple equation, or neglecting, or omitting, all the terms of it which involved in them any higher power of z than its ſimple power; which reduces the reſolution of all equations, of whatever orders, to the reſolution of a ſimple equation, or, rather, to the reſolution of ſeveral ſucceſſive ſimple equations, by which we make continual approaches to the true value of the root of the original equation. In this grand principle Sir Iſaac Newton's method and Mr. Raphſon's method perfectly agree; and, in finding the ſecond near value of x, or in making the firſt approximation to the true value of x, after having obtained, by conjecture, or trial, or in ſome other manner, the value of what has been here called a, or a firſt near value of x, or the root ſought, there is not the ſmalleſt difference between them. But in the inveſtigation of the third, and fourth, and other following near values of x, there is a little difference in their manner of proceeding, which the reader may be glad to ſee examined.

I ſhall

I shall therefore now compare the two methods together, in the case of a very easy equation, by which Sir Isaac Newton himself has thought proper to illustrate his method.

A COMPARISON BETWEEN SIR ISAAC NEWTON'S AND MR. RAPHSON'S METHODS OF RESOLVING NUMERAL EQUATIONS BY APPROXIMATION.

Art. XIII. SIR ISAAC NEWTON's method of resolving numeral equations by approximation, is explained by himself in his curious little Tract, intitled, *Analysis per Æquationes Numero Terminorum Infinitas*, (which was written in the year 1666, and communicated to Dr. Isaac Barrow, and to Mr. John Collins, and to other learned men of that time, in the year 1669,) by an example; which is as follows:

Art. XIV. Let it be required to resolve the cubick equation $x^3 - 2x = 5$.

Here, in the first place, it is easy to see that x is somewhat greater than 2, but much less than 3. For, if x is taken equal to 2, we shall have $2x = 4$, and $x^3 = 8$, and consequently $x^3 - 2x\ (= 8 - 4) = 4$; which is less than 5, or the true value of $x^3 - 2x$ in the proposed equation:

and, if x is taken equal to 3, we ſhall have $2x = 6$, and $x^3 = 27$, and conſequently $x^3 - 2x\ (= 27 - 6) = 21$; which is very much greater than 5, or the true value of $x^3 - 2x$ in the propoſed equation. Therefore the true value of x in that equation muſt be much leſs than 3, and a little greater than 2. Let it therefore be ſuppoſed to be equal to the quantity $2 + z$, in which z denotes the unknown quantity by which the true value of x exceeds 2. And let $2 + z$ be ſubſtituted, inſtead of x, in the propoſed equation $x^3 - 2x = 5$. This may be done as follows:

Since x is $= 2 + z$, we ſhall have

$x^3\ (= \overline{2}|^3 + 3 \times \overline{2}|^2 \times z + 3 \times 2 \times zz + z^3 = 8 + 3 \times 4 \times z + 3 \times 2zz + z^3) = 8 + 12z + 6zz + z^3$, and $2x\ (= 2 \times \overline{2 + z}) = 4 + 2z$, and conſequently $x^3 - 2x\ (= 8 + 12z + 6zz + z^3 - 4 - 2z) = 4 + 10z + 6zz + z^3$. But $x^3 - 2x$ is $= 5$. Therefore $4 + 10z + 6zz + z^3$ will alſo be $= 5$, and conſequently (ſubtracting 4 from both ſides,) $10z + 6zz + z^3$ will be $= 1$; and, (ſubtracting $6zz + z^3$ from both ſides,) $10z$ will be $= 1 - 6zz - z^3$. Therefore z will be $= \frac{1 - 6zz - z^3}{10} = \frac{1}{10} - \overline{\left|\frac{6zz + z^3}{10}\right.} = 0.1 \frac{-6zz - z^3}{10}$, that is, z is leſs than $\frac{1}{10}$, or 0.1, by the quantity $\frac{6zz + z^3}{10}$. Therefore x, or $2 + z$, is leſs than $2 + \frac{1}{10}$, or $2 + 0.1$, or 2.1, by the ſaid quantity $\frac{6zz + z^3}{10}$; which, on account of the ſmallneſs of z, (which is leſs than $\frac{1}{10}$,) will be a very

very ſmall quantity in compariſon of z, or of $\frac{r^{n-1}}{10}$, and, *a fortiori*, in compariſon of 2, and conſequently may be neglected. And therefore 2.1 will be a ſecond near value of x, or the root of the propoſed equation $x^3 - 2x = 5$, that will be a little greater than its true value, but nearer to it than any other number that conſiſts of only two places of figures.

Q. E. I.

This is the firſt ſtep of Sir Iſaac Newton's approximation to the root of the equation $x^3 - 2x = 5$, after the aſſumption of the number 2, by conjecture and trial, for its firſt near value. And in this firſt ſtep of the approximation Sir Iſaac Newton's and Mr. Raphſon's methods exactly co-incide.

Art. XV. But in the next ſtep of the approximation to the value x, in the ſaid equation $x^3 - 2x = 5$, the two methods are ſomewhat different from each other, though the number of new figures of the true value of x, that are exact in the next near values of it reſulting from both methods, is the ſame. The difference between the methods in this ſecond ſtage of the approximation is as follows :

Mr. Raphſon corrects the value of x, or the root of the original equation $x^3 - 2x = 5$, already found, to wit, 2.1, (and which is known to be ſomewhat greater than the truth,) by ſubtracting from it the unknown quantity by which it exceeds x, and which we may call v,

and ſubſtituting $2.1 - v$ inſtead of x in the ſaid original equation, $x^3 - 2x = 5$, whereby it is transformed into another cubick equation, in which v will be the only unknown quantity; and then he finds a near value of v by reſolving the ſaid transformed equation as if it were only a ſimple equation, or by neglecting the terms which involve the ſquare and cube of v, on account of their ſmallneſs, juſt as we before neglected the terms $6zz$ and z^3 in the foregoing transformed equation $10z + 6zz + z^3 = 1$ for the ſame reaſon. But Sir Iſaac Newton takes no further notice of the original equation $x^3 - 2x = 5$, till he has compleated the whole proceſs of his approximation; but, inſtead of the ſaid original equation, he conſiders the former transformed equation, $10z + 6zz + z^3 = 1$, which was derived from it, and inveſtigates the value of its root, z, to a greater degree of exactneſs than that to which it was before obtained. And this he does in the manner following:

Since it has been ſeen that z is leſs than 0.1, let the quantity by which 0.1 exceeds it be called v, ſo that z ſhall be $= 0.1 - v$; and let $0.1 - v$ be ſubſtituted, inſtead of z in the transformed equation $10z + 6zz + z^3 = 1$. This may be done as follows:

Since z is $= 0.1 - v$, we ſhall have

$$zz\ (= \overline{0.1 - v}^2) = 0.01 - 0.2v + vv,$$

and $z^3\ (= \overline{0.1 - v}^3 = 0.001 - 3 \times 0.01 \times v + 3 \times 0.1 \times vv - v^3) = 0.001 - 0.03v + 0.3vv - v^3$,

and

and $10z\ (= 10 \times 0.1 - v) = 1 - 10v$,

and $6zz = 0.06 - 1.2v + 6vv$,

and conſequently

$$\left\{\begin{matrix} 10z \\ +6zz \\ +\ z^3 \end{matrix}\right\} = \left\{\begin{matrix} 1.00 & - 10.00v & \\ +\ 0.06 & -\ 1.20v & +\ 6vv \\ +\ 0.001 & -\ 0.03v & +\ 0.3vv\ -\ v^3 \end{matrix}\right\}$$

$$= 1.061 - 11.23v + 6.3vv - v^3.$$

But $10z + 6zz + z^3$ is $= 1$.

Therefore $1.061 - 11.23v + 6.3vv - v^3$ will likewiſe be $= 1$. And conſequently (adding $11.23v$ to both ſides,) we ſhall have $1.061 + 6.3vv - v^3 = 1 + 11.23v$; and, (ſubtracting 1 from both ſides,) we ſhall have $0.061 + 6.3vv - v^3 = 11.23v$, and (neglecting $6.3vv$ and v^3 as inconſiderable in compariſon of 0.061 and $11.23v$) we ſhall have $0.061 = 11.23v$, or $11.23v = 0.061$; and conſequently (dividing both ſides by 11.23,) we ſhall have $(= \frac{0.061}{11.23} = 0\ 0054$. Therefore z, or $0.1 - v$, will be $(= 0.1 - 0.0054) = 0.0946$, and conſequently x, or $2 + z$, will be $(= 2 + 0.0946) = 2.0946$.

Q. E. I.

In this manner Sir Iſaac Newton finds the root of the propoſed equation $x^3 - 2x = 5$ to be equal to 2.0946, which is as near the truth as five figures can expreſs it.

Art. XVI. He then carries the inveſtigation one ſtep further, by which he obtains the value of π exact to nine places of figures; and for this purpoſe he proceeds in the manner following:

The laſt transformed equation was $11.23v = 0.061 + 6.3vv - v^3$; from which it follows that v is accurately equal to $\frac{0.061}{11.23} + \frac{6.3vv - v^3}{11.23}$, or $0.0054 + \frac{6.3vv - v^3}{11.23}$, which is greater than 0.0054 alone, becauſe $6.3vv$ is greater than v^3. Since, therefore, v is greater than 0.0054, let us ſuppoſe it to be $= 0.0054 + w$; and let this binomial quantity be ſubſtituted, inſtead of v, in the transformed equation $11.23v = 0.061 + 6.3vv - v^3$, or, rather, in the equation $11.23v - 6.3vv + v^3 = 0.061$, conſiſting of the ſame terms as the former, but in which the terms involving the unknown quantity v are all brought to the ſame ſide of the equation, and ranged according to the powers of v, beginning from its loweſt power, or the ſimple power of v. This may be done in the manner following:

Since v is $= 0.0054 + w$, we ſhall have

$$vv\ (=\overline{0.0054+w}^2 = \overline{0.0054}^2 + 2 \times 0.0054 \times w + w^2)$$
$$= 0.000,029,16 + 0.0108 \times w + w^2,$$
$$\text{and } v^3\ (=\overline{0.0054+w}^3 = \overline{0.0054}^3 + 3 \times \overline{0.0054}^2 \times w + 3 \times 0.0054 \times w^2 + w^3$$
$$= 0.000,000,157,464 + 3 \times 0.000,029,16 \times w + 0.0162 \times w^2 + w^3)$$

=

$= 0.000.000,157,464 + 0.000,087,48 \times w +$
$0.0162 \times w^2 + w^3,$

and $11.23vv (= 11.23 \times \overline{0.0054 + w}) = 0.060,642 +$
$11.23 \times w,$

and $6.3vv (= 6.3 \times \overline{0.000,029,16 + 0.0108 \times w + w^2})$
$= 0.000,183,708 + 0.068,04 \times w + 6.3ww;$

and consequently $11.23v - 6.3vv + v^3$ will be $=$

$$\left\{\begin{array}{l} 0.060,642 + 11.23 + w \\ -0.000,183,708 - 0.068,04 \times w - 6.3 \times ww \\ +0.000,000,157,464 + 0.000,087,48 \times w + 0.0162w^2 + w^3) \end{array}\right\}$$

$$= \left\{\begin{array}{l} 0.060,642,157,464 + 11.230,087,48 \times w + 0.0162w^2 + w^3 \\ -0.000,183,708 \quad - 0.068,04 \times w \quad -6.3 \times w^2 \end{array}\right\}$$

$= 0.060,458,449,464 + 11.162,047,48w - 6.2838w^2 + w^3.$

But $11.23v - 6.3vv + v^3$ is $= 0.061.$

Therefore $0.060,458,449,464 + 11.162,047,48 \times w - 6.2838 \times ww + w^3$ will likewiſe be $= 0.061$; and conſequently (ſubtracting 0.060,458,449,464 from both ſides,) $11.162,047,48 \times w - 6.2838ww + w^3$ will be $(= 0.061,000,000,000 - 0.060,458,449,464) = 0.000,541,550,536$; and (neglecting the terms $6.2838ww$ and w^3, as inconſiderable in compariſon of $11.162,047,48 \times w$,) we ſhall have $11.162,047,48 \times w = 0.000,541,550,536$, and conſequently $w (= \frac{0.000,541,550,536}{11.162,047,48} = 0.000,048,52$. Therefore v, or $0.0054 + w$, will be $(= 0.0054 + 0.000,048,52) = 0.005,448,52$, and z, or $0.1 - v$, will be $(= 0.100,000,00 - 0.005,448,58) = 0.094,551,48$, and

x, or

x, or $2 + z$, will be $(= 2 + 0.094,551,48) = 2.094,551,48$; that is, the root of the propoſed equation $x^3 - 2x = 5$ will be $= 2.094,551,48$.

Q. E. I.

This number 2.094,551,48 is exact in all the figures, as will be ſhewn in a ſubſequent Article.

Art. XVII. Having thus ſet forth Sir Iſaac Newton's method of inveſtigating the root of the propoſed equation $x^3 - 2x = 5$ to nine places of figures, we muſt now perform the ſame thing by Mr. Raphſon's method, in order to make a compariſon between the neceſſary operations of the two methods.

Now Mr. Raphſon's method of approximating further to the root of the equation $x^3 - 2x = 5$, after having found it to be equal to $2 + 0.1 - \frac{6zz - z^3}{10}$, or to be ſomewhat leſs than 2.1, is to put v for the unknown quantity by which it falls ſhort of 2.1, and then to ſubſtitute the reſidual quantity $2.1 - v$ in the terms of the original equation $x^3 - 2x = 5$, whereby the ſaid equation will be transformed into another cubick equation, in which v will be the only unknown quantity: and then he determines the value of v by reſolving the ſaid transformed equation as if it was a mere ſimple equation, or by neglecting the terms in which the ſquare or the cube of v occur. This may be done in the manner following:

Since

Since x is $= 2.1 - v$, we ſhall have

$xx\ (= \overline{2.1 - v}^2 = \overline{2.1}^2 - 2 \times 2.1 \times v + \&c)$

$= 4.41 - 4.2v + \&c,$

and $x^3\ (= \overline{2.1 - v}^3 = \overline{2.1}^3 - 3 \times \overline{2.1}^2 \times v + \&c$

$= 9.261 - 3 \times 4.41 \times v + \&c)$

$= 9.261 - 13.23 \times v + \&c,$

and $2x\ (= 2 \times \overline{2.1 - v}) = 4.2 - 2v,$

and conſequently $x^3 - 2x\ (= 9.261 - 13.23 \times v + \&c$

$- 4.2 + 2v) = 5.061 - 11.23 \times v \&c.$

But $x^3 - 2x$ is $= 5.$

Therefore $5.061 - 11.23 \times v$ &c, will likewiſe be $= 5$, and conſequently (adding $11.23 \times v$ to both ſides,) we ſhall have $5.061 = 5 + 11.23 \times v$, and (ſubtracting 5 from both ſides,) we ſhall have $11.23 \times v = 0.061$, and conſequently $v\left(= \frac{0.061}{11.23}\right) = 0.0054$. Therefore x, or $2.1 - v$, will be $(= 2.1 - 0.0054) = 2.0946$; or 2.0946 will be a third near value of the root of the propoſed equation $x^3 - 2x = 5$.

Q. E. I.

This third near value of x is the very ſame with the third near value of it obtained above, in Art. 15, by Sir Iſaac Newton's method.

Art. XVIII. In this ſtep of the approximation, by which we obtain the number 2.0946 for the third near value of the root of the propoſed equation $x^3 - 2x = 5$, the principal

cipal difference between the two methods ſeems to conſiſt in this, to wit, that by Mr. Raphſon's method we are obliged to raiſe the two firſt terms of the powers of the compound quantity $2.1 - v$, and conſequently to raiſe the powers of the number 2.1, which conſiſts of two figures; whereas in Sir Iſaac Newton's method of proceeding, we had occaſion only to raiſe the powers of the compound quantity $0.1 - v$, and conſequently to raiſe the powers of the number 0.1, which conſiſts of only one figure; which is ſomewhat eaſier than to raiſe the powers of 2.1. But both operations are ſo eaſy, that the difference of the labour of performing them is hardly worth conſidering. And, with reſpect to the ſimplicity of conception in the two methods, Mr. Raphſon's method ſeems to be preferable to Sir Iſaac Newton's; becauſe the former always refers to the original equation $x^3 - 2x = 5$, whereas the latter method refers to the preceeding transformed equation $10z + 6zz + z^3 = 1$, which has more terms and larger co-efficients than the original equation $x^3 - 2x = 5$.

Art. XIX. But in the next ſtep of the approximation by Mr. Raphſon's method, we ſhall find the labour of raiſing the powers of the value of x already found, to wit, the powers of 2.0946, to be conſiderably greater than that of raiſing the powers of the laſt preceeding ſupplement of it according to Sir Iſaac Newton's method, that ſupplement being only the decimal fraction 0.0054, in which there are only two ſignificant figures. This will appear by performing this ſtep of the approximation by Mr. Raphſon's method; which may be done as follows:

Art.

Art. XX. The laſt near value we found for x, or the root of the equation $x^3 - 2x = 5$, by Mr Raphſon's method, was 2.0946. Now this near value of x is greater than its true value. For, if we ſuppoſe x to be $= 2.0946$, we ſhall have $x^3 (= \overline{2.0946}^3) = 9.189,741,550,536$, and $2x (= 2 \times 2.0946) = 4.1892$, and conſequently $x^3 - 2x$ $(= 9.189,741,550,536 - 4.1892) + 5.000,541,550,536$; which is greater than 5, or the abſolute term of the equation $x^3 - 2x = 5$: and conſequently 2.0946 muſt be greater than the true value of the root of the ſaid equation.

We will therefore ſuppoſe x to be $= 2.0946 - w$, and ſubſtitute this reſidual quantity inſtead of x in the terms of the equation $x^3 - 2x = 5$.

Now, ſince x is $= 2.0946 - w$, we ſhall have

$xx (= \overline{2.0946 - w}^2 = \overline{2.0946}^2 - 2 \times 2.0946 \times w + \&c)$
$= 4.387,349,16 - 4.1892 \times w + \&c,$
and $x^3 (= \overline{2.0946 - w}^3 = \overline{2.0946}^3 - 3 \times \overline{2.0946}^2 \times w + \&c$
$= 9.189,741,550,536 - 3 \times 4.387,349,16 \times w + \&c)$
$= 9.189,741,550,536 - 13.162,047,48 \times w + \&c,$
and $2x (= 2 \times \overline{2.0946 - w}) = 4.1892 - 2w,$

and conſequently $x^3 - 2x =$

$$\left\{\begin{array}{rrl} 9.189,741,550,536 & - & 13.162,047,48 \times w + \&c \\ - 4.189,2 & + & 2.000,000,00 \times w \end{array}\right\}$$

$= 5.000,541,550,536 - 11.162,047,48 \times w + \&c.$

But

But $x^3 - 2x$ is $= 5$.

Therefore 5,000,541,550,536 — 11.162,047,48 × w + &c, will be $= 5$; and conſequently (adding 11.162,047,48 × w to both ſides,) we ſhall have 5.000,541,550,536 $= 5 +$ 11.162,047,48 × w, and (ſubtracting 5 from both ſides,) 0.000,541,550,536 $=$ 11.162,047,48 × w, or 11.162,047,48 × w = 0.000,541,550,536. Therefore w will be $(= \frac{0.000,541,550,536}{11.162,047,48})$ 0.000,048,52; and conſequently x, or 2.0946 − w, will be (= 2.094,600,00 — 0.000,048,52) = 2.094,551,48. Therefore 2.094,551,48 will be a fourth near value of x, or the root of the propoſed equation $x^3 - 2x = 5$.

Q. E. I.

This fourth near value of x is the very ſame with the fourth near value of it obtained above, in Art. 16, by Sir Iſaac Newton's method.

Art. XXI. In this laſt ſtage of Mr. Raphſon's approximation to the root of the propoſed equation $x^3 - 2x = 5$, we have been obliged to raiſe the powers of the number 2.0946, which conſiſts of five places of figures; whereas in Sir Iſaac Newton's way of proceeding we only raiſed the powers of the decimal fraction 0.0054, which contains only two ſignificant figures. But then in that way of proceeding we were obliged to multiply v, or 0.0054 + w, into 11.23, and vv, or 0.000,029,16 + 0.0108 × w + w^2, into 6.3; whereas in Mr. Raphſon's way of proceeding we have only to multiply x, or 2.0946 — w, into

the very ſimple co-efficient 2. So that, upon the whole, the difference of the labour of computation in the two methods is not very conſiderable, though it is rather leſs in Sir Iſaac Newton's method than in Mr. Raphſon's. But in point of ſimplicity of conception Mr. Raphſon's method ſeems much ſuperiour to Sir Iſaac's, becauſe it never loſes ſight of the original equation $x^3 - 2x = 5$, which is to be reſolved.

And, further, we may obſerve, in favour of Mr. Raphſon's method, that it never requires us to raiſe any more than the two firſt terms of the binomial and reſidual quantities $2 + z$, and $2.1 - v$, and $2.0946 - w$, which are ſubſtituted inſtead of x in the original equation $x^3 - 2x = 5$; whereas in Sir Iſaac Newton's method it is neceſſary to raiſe the other terms of the binomial and reſidual quantities $2 + z$, and $0.1 - v$, and $0.0054 + w$; which increaſes the number and intricacy of the operations of the inveſtigation. And therefore, upon the whole, I conſider Mr. Raphſon's method of approximating to the values of the roots of ſuch equations as preferable to Sir Iſaac Newton's.

A PROOF OF THE EXACTNESS OF THE NUMBER 2.094,551,48, THAT HAS BEEN FOUND BY THE FOREGOING METHODS OF APPROXIMATION FOR THE ROOT OF THE EQUATION $x^3 - 2x = 5$.

Art. XXII. IT remains that we prove the work to have been rightly performed, or that we ſhew that the laſt

laſt number 2.094,551,48, obtained by both theſe methods, is a very near value of the root x of the propoſed equation $x^3 - 2x = 5$, and that we determine to how many figures it is exact.

Now the plaineſt and beſt method of doing this is to ſubſtitute the number 2.094,551,48, inſtead of x, in the compound quantity $x^3 - 2x$, in order to diſcover whether the quantity reſulting from this ſubſtitution will be greater, or leſs, than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$: and, if it ſhall appear that the ſaid reſult is greater than 5, we may conclude that the ſaid number 2.094,551,48 is greater than the true value of x in the ſaid equation; and, if it ſhall appear that the ſaid reſult is leſs than 5, we may conclude that the ſaid number is leſs than the true value of x. And, when this has been thus diſcovered, we muſt, in the next place, endeavour to determine to how many figures this number 2.094,551,48 co-incides with the more accurate value of x: and, for this purpoſe, we muſt, if this number be leſs than x, increaſe it by the addition of an unit in the laſt place of figures; and, if it be greater than x, we muſt diminiſh it by the ſame ſmall quantity, and then ſubſtitute the new number thereby obtained, to wit, 2.094,551,49, or 2.094,551,47, inſtead of x, in the compound quantity $x^3 - 2x$. And, if it ſhall appear that the value of that compound quantity reſulting from ſuch ſubſtitution is greater, or leſs, than 5, we may conclude that the number 2.094,551,49, or 2.094,551,47, is accordingly greater, or leſs, than the true value of x, in the equation $x^3 - 2x = 5$, and conſequently that the ſaid true value is of an intermediate

termediate magnitude between 2.094,551,49 and 2.094,551,48, or between 2.094,551,48 and 2.094,551,47.

Now, if we take $x = 2.094,551,48$, we ſhall have

$xx = 4.387,145,902,370,190,4,$

and $x^3 = 9.189,102,942,785,417,810,201,792,$

and $2x = 4.189,102,96,$

and conſequently

$x^3 - 2x = 4.999,999,982,785,417,810,201,792$; which number is ſomewhat leſs than 5, or the abſolute term of the propoſed equation $x^3 - 2x = 5$. Therefore 2.094,551,48 muſt be ſomewhat leſs than the true value of x in the ſaid equation.

Secondly, ſince x is greater than 2.094,551,48, we muſt now compare it with 2.094,551,49, by ſubſtituting that number inſtead of it in the compound quantity $x^3 - 2x$.

Now, if x is taken = 2.094,551,49, or 2.094,551,48 + 0.000,000,01, we ſhall have x^3 (= $\overline{2.094,551,48}^3$ + 3 × $\overline{2.094,551,48}^2$ × $\overline{0.000,000,01}$ + 3 × 2.094,551,48 × $\overline{0.000,000,01}^2$ + $\overline{0.000,000,01}^3$ = 9.189,102,942, &c, + 3 × 4.387,145,902, &c × 0.000,000,01 + 0.000,000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 13.161,437,706, &c × 0.000,000,01 + 0.000,000,000, &c + 0.000,000,000, &c = 9.189,102,942, &c + 0.000,000,131, &c + 0.000,000,000, &c + 0.000,000,000, &c) = 9.189,103,073, &c; and $2x$ (= 2 × 2.094,

551,49) = 4.189,102,98; and consequently $x^3 - 2x$ (= 9.189,103,07, &c — 4.189,102,98) = 5.000,000,09, &c; which is greater than 5. Therefore 2.094,551,49 must be greater than the true value of x in the equation $x^3 - 2x = 5$.

But it has been shewn that 2.094,551,48 is less than the said true value.

Therefore the true value of x in the equation $x^3 - 2x = 5$, will be of an intermediate magnitude between 2.094,551,48 and 2.094,551,49; and consequently all the figures of the number 2.094,551,48, which we found by the foregoing processes of Sir Isaac Newton's and Mr. Raphson's methods of approximation for a fourth near value of the root of the equation $x^3 - 2x = 5$, are exact.

Q. E. D.

OF THE DIFFICULTY OF FINDING a, OR THE FIRST NEAR VALUE OF THE ROOT OF AN AFFECTED EQUATION, IN CERTAIN CASES.

Art. XXIII. THERE is another difficulty that occurs sometimes in resolving high equations by approximation, whether

whether by Sir Iſaac Newton's method or by Mr. Raphſon's; which indeed are ſubſtantially the ſame. The difficulty I mean, is that of finding the firſt near value of the root ſought (which we have called a in this diſcourſe,) to one, or two places of figures, in order to make it the baſis of a further approximation to the true value of the root by either of theſe methods of approximation. Now, when the equation is known to have but one root, that is, but one real and affirmative root, (for all other roots are not worth conſidering,) this difficulty will not be great; becauſe it will always be eaſy to find a tolerably near value of the root by conjectures and trials, and particularly by ſuppoſing x, or the root of the propoſed equation, firſt to be equal to 1, and 2dly, to be $= 10$, and 3dly, to be equal to ſome ſhort intermediate number conſiſting of only one figure, or, if the root appears to be greater than 10, by ſuppoſing it to be equal to 100, or 1000, and afterwards ſuppoſing it to be equal to ſome ſhort intermediate number conſiſting of two figures; as was done above in Art. 5, in finding the firſt near value of x in the equation $x^5 + 7x^4 + 20x^3 + 155xx = 10{,}000$. But, when the equation conſiſts of terms connected together partly by the ſign $+$, and partly by the ſign $-$, and conſequently it may, for aught we know to the contrary, have two, or three, or four, or more real and affirmative roots, which may be of very different magnitudes, the aforeſaid method of conjectures and trials (though by no means uſeleſs,) is leſs expeditious and ſatisfactory in aſſiſting us to find the firſt near value of one of the roots than in the former caſe; and we are often puzzled to know which of the roots it would be moſt expedient to begin to inveſti-

gate. Now, in moſt of theſe caſes, I believe, it will be adviſeable to begin by inveſtigating the leaſt root, and for that purpoſe to expunge from the equation all the terms that have the ſign — prefixed to them, and to find, to about two places, or, at moſt, to three places, of figures, the root of the remaining equation. For this root will always be leſs than the leaſt root of the original equation, if it really has (as it appears to have,) more than one real and affirmative root; or it will be leſs than the only root of the original equation, if (notwithſtanding the appearances to the contrary,) it really has but one root. When the root of this ſecond, or curtailed, equation, has been diſcovered, it may be called a, and made the ground-work of an approximation to the leaſt root of the original equation, and the binomial quantity $a + z$ may be ſubſtituted in the original equation inſtead of x, and the transformed equation thence ariſing may be reſolved as if it was a mere ſimple equation, agreeably to Mr. Raphſon's method of approximation; and the value of z thereby obtained, being added to a, will give us a known value of $a + z$, or a ſecond near value of the leaſt, or the only, root of the propoſed equation: after which we may proceed to find the ſaid leaſt, or only, root of the propoſed equation by a further proſecution of Mr. Raphſon's method of approximation above-deſcribed. This method of finding a firſt near value, a, of the leaſt root of a propoſed equation that ſeems to have more than one real and affirmative root, is explained more at length in the third volume of the Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in my Diſcourſe on the Reverſion of Infinite Serieſes publiſhed in that Volume;

lume; to which I refer the reader. See the ſaid 3d Volume, pages 724, 725, 726, 727, &c, - - - to page 761. And, with this improvement of it in the caſe of equations that have, or ſeem to have, more than one real and poſitive root, I believe it may ſafely be affirmed that Mr. Raphſon's Method of Reſolving Affected Equations is the beſt general method of effecting that purpoſe in all equations above quadraticks that has hitherto been diſcovered.

End of the Obſervations on Mr. Raphſon's *Method of Reſolving Affected Equations by Approximation.*

AN

EXPLICATION

OF

SIMON STEVIN's GENERAL RULE,

TO EXTRACT ONE ROOT OUT OF ANY POSSIBLE EQUATION IN NUMBERS, EITHER EXACTLY, OR VERY NEARLY TRUE.

By JOHN KERSEY.

BEING THE 10TH CHAPTER OF THE SECOND BOOK OF MR. KERSEY'S ELEMENTS OF ALGEBRA.

Art. I. EQUATIONS falling under any of the forms in the fourteenth and fifteenth chapters of the firſt book of theſe Elements, are capable (as hath there been ſhewn) of perfect reſolutions in numbers; viz. the value of the root or roots ſought in any of thoſe equations may be found out and expreſſed exactly, either by ſome rational or irrational number or numbers; but the perfect reſolution of all manner of compound equations in numbers, I have not found in any author: and ſince an expoſition of the general method of Vieta, the rules of Huddenius and others to that purpoſe, would make a large treatiſe, and after all leave the curious Analyſt diſſatisfied, I ſhall not clogg theſe Elements

ments with a tedious diſcourſe upon thoſe difficult rules, (which at the beſt are exceeding tedious in operation, and ſome of them uncertain too,) but rather purſue my firſt deſign, which was to explain fundamentals, and ſuch rules as are certain and moſt important in this profound art. However, I ſhall lead the induſtrious learner a few ſteps farther in order to his underſtanding the reſolution of all manner of compound equations in numbers, and in this Chapter ſhall explain Simon Stevin's General Rule, which, with the help of the rules in the following eleventh Chapter, will diſcover all the roots of any poſſible equation in numbers, either exactly, if they be rational, or very nearly true, if irrational.

QUESTION I.

IF - - - $aaa + 26a$ is $= 40188$, what is the number a?

RESOLUTION.

THIS equation not falling under any of the three forms in Sect. I. chap. XV. book I. cannot be reſolved by any of the canons in that chapter, and therefore according to Simon Stevin's general method I ſearch out the number a by trials, thus, viz.

1. I ſuppoſe - - - - $a = 1$
Thence it follows that - - $aaa = 1$
And - - - - - $26a = 26$
Therefore - - $aaa + 26a = 27$;

Which 27 ought to have been 40188, but it's too little; whereby I find that, by ſuppoſing a to be 1, I did not hit upon the true number a; and therefore I make another trial, in like manner as before, viz.

2. I ſuppoſe - - - - $a = 10$
Thence it follows that - - $aaa = 1000$
And - - - - $26a = 260$
Therefore - - $aaa + 26a = 1260$

Which 1260 being yet too little, I make a third trial, viz.

3. I ſuppoſe - - - $a = 100$
Thence it follows, that $aaa + 26a = 1,002,600$

Which 1,002,600 exceeds the juſt reſult, or abſolute number 40,188 in the latter part of the equation firſt propoſed, and therefore the true number a is leſs than 100; but the ſecond trial ſhews it to be greater than 10, and therefore the whole number which expreſſeth the exact, or, at leaſt, part of the value of a, muſt neceſſarily conſiſt of two characters, and conſequently the firſt (towards the left hand) muſt be one of theſe nine, 1, 2, 3, 4, 5, 6, 7, 8, 9. But, becauſe, by the ſecond inquiry, 10 was found too little,

little, I now make trial with 2 for the firſt figure of the root a, viz.

4. I ſuppoſe - - - - $a = 20$
Thence - - $aaa + 26a = 8520$

Which reſult 8520 being yet leſs than the juſt reſult 40,188, I make trial again, viz.

5. I ſuppoſe - - - - $a = 30$
Thence - - $aaa + 26a = 27780$

Which is yet too little; therefore,

6. I ſuppoſe - - - - $a = 40$
Thence - - - $aaa + 26a = 65,040$

Which 65,040 being greater than 40,188, it ſhews me that the true root or value of a is leſs than 40; but by the fifth trial it is greater than 30, and conſequently the firſt figure of the root is 3.

Now the ſecond character of the root muſt neceſſarily be one of theſe, viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and becauſe it hath been diſcovered that the true value of the root a is greater than 30, the ſecond character cannot be 0: I therefore make trial with 1, and ſuppoſe $a = 31$, which proving too little, I make trial with 32, 33, 34, &c. ſeverally, in like manner as before, and at length I find 34 to be the true number a ſought, by which the equation propoſed may be expounded; for, if $a = 34$, then conſequently $aaa + 26a = 40,188$.

Art.

Art. II. But if after trials made (as before) the value of a, the root ſought, happens to fall between two whole numbers that differ by unity; then trials are to be made with the leſſer whole number increaſed with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of a in ſome mixt number conſiſting of a whole number and ſome certain tenth parts of an unit: But if the ſaid value of a happens not to be expreſſed exactly by the ſaid leſſer whole number increaſed with certain tenth parts, then you are to make trials with the ſaid leſſer whole number increaſed with a decimal fraction having for its numerator a number greater than 10, but leſs than 100; and for its denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the root a when its fractional part is exactly equal to ſome decimal fraction, or elſe approach infinitely near to the ſaid exact value when it is irrational or ſurd, as in this following;

QUESTION II.

IF $\ldots aaaa + 50a$ is $= 184,638.6801$; (or, $184,638\frac{6801}{10000}$;) what is the number a?

RESOLUTION.

FIRST, I ſuppoſe $a = 1$; but this proving too little, I put $a = 10$; this alſo proving too little, I aſſume $a = 100$, which after trial I find to be greater than the true number a, and conſequently the number a falls between 10 and 100;

100; then, making trial with 20, I find it too little; but making trial with 30, I find this too great; and therefore the true root a falls between 20 and 30. Again, making trial with 21, I find it too great: but 20 was before found too little; therefore the true root a is between 20 and 21; then I make trial with 20.1, (that is, $20\frac{1}{10}$,) 20.2; 20.3, &c. and at length find 20.7 to be the true number a fought; for if $a = 20.7$ (that is, $20\frac{7}{10}$,) it will make $aaaa + 50a = 184,638.6801$ the equation propoſed *.

But if 20.7 had proved too little, and 20.8 too great, then trials muſt have been made with 20.71, (that is, $20\frac{71}{100}$;) 20.72; 20.73, &c. In like manner, if 20.7 had been too little, but 20.71 (that is, $20\frac{71}{100}$) too great, then trials muſt have been made with 20.701, (that is, $20\frac{701}{1000}$;) 20.702; 20,703, &c. This will be partly exerciſed in reſolving the equation in this following—

QUESTION III.

IF $- - - aaa + 20qa = 1954$, what is the number a?

Anſ. $- - a = 8.308$, &c. found out by trials, as before.

Art. III. When the value of (a) the required root of

* This method of proceeding is very convenient for finding the two firſt figures of the value of a; but then it will be expedient to have recourſe to Mr. Raphſon's method of approximation, which will uſually double, or nearly double, the number of figures in the value of a at every operation, See his *Analyſis Æquationum Univerſalis*.

an equation happens to be leſs than unity, then trial is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now ſuppoſe .1 (that is, $\frac{1}{10}$) be too great, but .01 (that is, $\frac{1}{100}$) too little; then trial muſt be made with .02 | .03 | .04 | &c. until you have found out the greateſt figure that muſt ſtand in the ſecond place of the decimal fraction expreſſing the root ſought; ſuppoſing then ſuch figure to be found 8, viz, that .08 (or $\frac{8}{100}$) is leſs, but .09 (or $\frac{9}{100}$) is greater than the root, trial muſt be made with .081, (that is, $\frac{81}{1000}$,) .082 | .083 | &c. as in this following—

QUESTION IV.

IF - - $aaa + 3240a$ is $= 269$, what is the number a?

Anſw. - a is $= .083$, &c. that is, $\frac{83}{1000}$, &c.

Art. IV. The preceeding examples may ſuffice to ſhew the uſe of this general method when all the terms of the unknown part of an equation are affirmative, (viz. when + is prefixed to each term,) in which caſe there is but one affirmative root; in the ſearch whereof by trials (as before) if the numbers aſſumed ſeverally for the value of the root ſought do aſcend, or become greater and greater, then the abſolute numbers reſulting from thoſe aſſumed values will likewiſe aſcend; and contrary, if the aſſumed roots do deſcend from a greater to a leſs, the reſults will likewiſe grow leſs and leſs: whence, by comparing an abſolute number reſulting from an aſſumed root with the juſt abſolute number of the equation propoſed, you may certainly know

(if

(if the ſaid reſult and juſt abſolute be not equal to one another) whether you are to take a number greater or leſs than that laſt before aſſumed.

But when the unknown part of an equation conſiſts of affirmative and negative terms mingled one with another, then the ſearch by trials will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed root muſt be taken greater or leſs than that which was taken next before. Moreover, a compound equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative roots as there be unities in the index of the higheſt unknown power, viz. a cubical equation may be ſo conſtituted that it ſhall have three different affirmative roots, a biquadratick equation four ſeveral roots; and ſo of higher equations, as will be ſhewn in the following Chap. II. But, in what manner ſoever any poſſible equation is conſtituted in rational numbers, this general method will always find out one affirmative root, either exactly true, or at leaſt very near the truth; as will farther appear by the following queſtions:

QUESTION V.

IF - - - $aaa - 22aa + 157a = 360$, what is the number a?

RESOLUTION.

1. I SUPPOSE - - - - - $a = 1$

Thence it follows that $aaa - 22aa + 157a = 136$

Which

Which 136 is leſs than the juſt abſolute number 360, and therefore I make another trial, viz.

2. I ſuppoſe - - - - $a = 10$

Thence it follows that $aaa - 22aa + 157a = 370$

Which 370 exceeds the juſt abſolute number 360, and therefore I conclude there is one affirmative value of a, (either rational or irrational) between 1 and 10, which value, after trials made with 2, 3, 4, 5, I find to be 5; this will conſtitute the equation propoſed; for if $a = 5$, then $aaa - 22aa + 157a$ will exactly make 360.

But there are two other roots or values of a, to wit, 8 and 9, each of which will likewiſe conſtitute the equation firſt propoſed; but how they are found out will be ſhewn in Sect. 9. of the following Chap. 11.

QUESTION VI.

IF - - - - $3200a - aaa = 46,577$ (juſt,) what is the number a?

RESOLUTION.

1. I SUPPOSE - - $a = 1$

Thence - $3200a - aaa = 3199$ (leſs than juſt.)

2. I ſuppoſe - - $a = 10$

Thence - $3200a - aaa = 31000$ (leſs than juſt.)

3. I

3. I ſuppoſe - - $a = 100$
Thence - $3200a - aaa = -680{,}000$ (leſs than juſt.)

Now becauſe the ſecond reſult (or abſolute number) + 31000 is affirmative, and the laſt reſult — 680,000 is negative, I make trials with numbers between 10 and 100 for the value of a; for, if the equation propoſed be poſſible, before the affirmative reſults fall off to negatives, there will be a root, or value of a, producing an affirmative reſult either exactly equal, or very near to the juſt reſult 46,577; therefore,

4. I ſuppoſe - - $a = 20$
Thence - $3200a - aaa = 56{,}000$ (greater than juſt.)

Now becauſe by taking 20 for the value of a, the reſult 56,000 exceeds the juſt reſult 46,577; but by taking 10 for a, the reſult 31000 happened to be leſs than the ſaid 46577; it ſhews that there is one affirmative root or value of a between 10 and 20, which root, after trials made with intermediate numbers (as in former examples) will be found 15.7, &c. Moreover, becauſe, by ſuppoſing $a = 20$, the reſult 56,000 happened to exceed the juſt reſult 46,577, but, by putting $a = 100$, the reſult — 680000 proved to be leſs than the ſame 46,577, it ſhews there is an affirmative value of a between 20 and 100, which value after trials made will be found 47: ſo that there are two affirmative roots or values of a found out, to wit, 15.7, &c. (or $15\frac{7}{10}$, &c.) and 47; the former of which will nearly, and the latter exactly, conſtitute the equation propoſed.

Art.

Art. V. Florimond de Beaune in the latter of two ſmall treatiſes printed in 1659, concerning the nature, conſtitution and limits of equations *, ſhews how to find out limits within which the roots of all compound equations not aſcending above the biquadratick kind are confined; which limits, when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the trials in the general method before delivered: As, in the laſt Example, where

The equation propoſed was $3200a - aaa = 46,577$

Firſt, becauſe *aaa* muſt be ſubtracted from $3200a$ and leave a remainder equal to 46,577, it preſuppoſeth	- - *aaa* to be $\sqsupset 3200a$
Therefore by dividing each part by *a*, - -	- - - $aa \sqsupset 3200$
And by extracting the ſquare root out of each part, it follows that - - -	- - - $a \sqsupset 56.5$, &c.
Again, from the equation propoſed, by tranſpoſition it is evident that - - -	$3200a - 46,577 = aaa$
Whence it is alſo manifeſt	- - $3200a \sqsubset 46,577$
And conſequently, by dividing each part by 3200,	- - $a \sqsubset 14.5$, &c.

* This tract of M. de Beaune is very clear and very uſeful. It is printed with Mr. Schooten's Comment on Des Cartes's Geometry.

Thus

Thus it is found that the value of *a* the root ſought is greater than 14.5, &c. but leſs than 56.5, &c. and therefore trials according to the general method aforeſaid need not be made with any numbers that are not within thoſe limits.

From the premiſes it is evident, that this general method finds not a perfect root of an equation, unleſs ſuch root be a whole number, or elſe a fraction exactly equal to ſome decimal fraction; or laſtly, a mixed number compoſed of a whole number and a perfect decimal fraction.

Note. When the co-efficients or known numbers multiplied into any of the unknown powers under the higheſt, (which muſt have no co-efficient but unity,) are vulgar (not decimal) fractions, or mixt numbers whoſe fractional parts are vulgar fractions; likewiſe, when the abſolute number that ſolely poſſeſſeth the latter part of the equation propoſed is a vulgar fraction, or a mixt number whoſe fractional part is a vulgar fraction; all thoſe vulgar fractions muſt be reduced to decimal fractions, or elſe the equation muſt be reduced to another equation in integers (by Sect. 7. in the following Chap. 11.) before you enter upon the reſolution by trials as aforeſaid.

End of the 10*th Chapter of the Second Book of Mr.* Kerſey's *Elements of Algebra.*

A REMARK ON AN ERROR IN THE REASONING OF THE LATE LEARNED FRENCH MATHEMATICIAN, MONSIEUR CLAIRAUT, IN THAT PART OF HIS *ELEMENTS OF ALGEBRA* IN WHICH HE ENDEAVOURS TO PROVE THE RULES OF MULTIPLICATION LAID DOWN BY WRITERS ON ALGEBRA CONCERNING NEGATIVE QUANTITIES.

THERE is one writer of Algebra who has treated the ſubject with uncommon care and elegance, and, throughout the firſt part of his work, with great perſpicuity likewiſe. I mean Monſieur Clairaut of the French Academy of Sciences, in his *Elémens d'Algébre* publiſhed at Paris in the year 1749. But even this writer has been betrayed into a moſt remarkable inſtance of falſe reaſoning, by his deſire of explaining to his readers that which in it's nature is not capable of being explained or underſtood, to wit, the propoſition "*that, if a negative quantity be multiplied by a negative quantity, the product will be an affirmative quantity.*" This propoſition he has attempted to demonſtrate in the 60th Section, or Article, of his book, page 73, in the following manner: He had already in Art. iii, page 4, explained the ſign — to ſignify the ſubtraction of the quantity to which it is prefixed from the quantity which immediately preceeds it, in theſe words: "*Prenant le caractere* —, *qui ſe prononce* moins, *pour faire reſſouvenir que la*

la quantité qu'il précède doit être retranchée de celle qu'il ſuit." And in Articles xlii, xliii, xliv, xlv, xlvi, pages 48, 49, 50, 51, 52, 53, 54, he explains the manner of multiplying compound quantities, (which he calls *des quantités complexes ou polynomes, ou quantités compoſées de pluſieurs termes,*) by each other, and exemplifies it by the multiplication of the compound quantity $2a^3e^2 - 5a^4b + 6a^5$ by the compound quantity $3ab^2 - 4bcd$. And thus far he reaſons juſtly and clearly, uſing the ſign — only in the ſenſe of his firſt definition of it in Art. iii, page 4, as the ſign of the ſubtraction of the quantity to which it is prefixed from that which goes before it, in which firſt part of his book he ſeems to ſuppoſe all along that the quantities to be ſubtracted are leſs than thoſe from which they are to be ſubtracted. But then, when he has thus ſhewn clearly that, in the multiplication of compound quantities by compound quantities, the rule holds good, that — into — gives +, or that thoſe members of the product of the two compound quantities which ariſe from the multiplication of any two members of the compound quantities themſelves that are both marked with the ſign —, will be marked with the ſign +, he endeavours, in Art. lx, page 73, to extend this concluſion to the product of two independent, or ſeparate, negative quantities, or quantities which are marked with the ſign — without being preceeded by any other quantities from which they are ſubtracted. This lxth Article is in theſe words:

Pour nous aſſurer que la multiplication de — par — doit toûjours donner + au produit, voyons quelle lumière nous pouvons tirer de la méthode générale des multiplications donnée dans

l'Art. xlv. Suivant cette méthode on voit très clairement que le produit d'une quantité telle que a — b *par une autre* c — d *doit être* ac — bc — ad + bd ; *et on voit par conſéquent en même temps que le terme* bd, *qui eſt venu par la multiplication de* b *et de* d, *a le ſigne* +, *tandis que ſes produiſants* b *et* d *ont le ſigne* —. *Il ne reſte donc plus qu' à ſçavoir ſi, lorſque deux quantités negatives, telles que* b *et* d, *ne ſeront précédées d'aucune quantité poſitive, leur produit ſera encore* + bd. *Or c'eſt ce dont il eſt facile de reconnoitre la vérité, puiſque la méthode par laquelle on a découvert que le produit de* a — b *par* c — d *étoit* ac — bc — ad + bd, *ne ſpécifiant aucune grandeur particuliére ni à* a *ni à* c, *doit avoir encore lieu lorſque ces quantités ſont égales à zéro. Or, en ce cas, le produit* ac —bc,—ad+bd *ſe réduit à* +bd. *Donc* —b × —d *eſt* = +bd.

Now in theſe words there is an obvious fallacy. For, though *a* and *c* are not particular numbers, or quantities which have a fixt relation to *b* and *d*, but they may be of many different magnitudes with reſpect to *b* and *d*, yet they muſt always be ſuppoſed to be *greater* than *b* and *d* reſpectively. For otherwiſe it will not be poſſible to ſubtract *b* and *d* from them, ſo as to produce the quantities *a* — *b* and *c* — *d* ; the author not having hitherto given us any other idea of the ſign — but that of its denoting the ſubtraction of the quantities to which it is prefixed from thoſe which go before them ; in order to which it is neceſſary, and is conſtantly ſuppoſed in all the foregoing part of the book, that the ſaid quantities, to which the ſaid ſign is prefixed, ſhould be leſs than thoſe that go before them. It follows therefore that *a* and *c* can never be ſuppoſed to be either equal to, or leſs than, *b* and *d*, reſpectively, though

though they may be of any magnitudes that are greater than b and d. It is not therefore poſſible for a and c to become equal to o, as the author ſuppoſes; and conſequently the compound quantity $ac - bc - ad + bd$, (which is the product of the multiplication of $a - b$ and $c - d$ into each other) can never become equal to $+ bd$ by the vaniſhing, or annihilation, of it's three firſt members ac, bc, ad, as the author concludes from the ſaid falſe ſuppoſition that a and c may be taken equal to o.

The author on this occaſion ſeems to have forgot his own definition of the ſign —, by which he made it to be a mark of the ſubtraction of the quantity to which it is prefixed from the quantity that goes before it; from which definition it is plain that the ſaid ſign always ſuppoſes the exiſtence of two different quantities, of which the one is to be ſubtracted from the other, and conſequently that it can have no meaning when applied to a ſingle quantity, as b or d, independently of ſome other and greater quantity, as a or c, from which it is to be ſubtracted. There cannot therefore exiſt any ſuch quantities as $-b$, or $-d$; and conſequently no propoſitions concerning them can be either true or falſe. Conſequently the propoſition which the author there endeavours to demonſtrate, to wit, "that $-b \times -d$ is equal to $+bd$, when $-b$ and $-d$ are not preceeded by two greater quantities a and c from which they are ſubtracted, but are conſidered as ſingle and independent quantities," is ſo far from being *true* that it is not even *intelligible* according to the only idea of the ſign — which the author has given us in all the preceeding part of the book.

Such

Such is the obſcurity and confuſion in which even this very able writer (who not only had a great deal of genius and ſagacity in the mathematical ſciences, but alſo an elegant taſte in his manner of treating them,) has involved himſelf by his attempt to explain the nature and properties of negative quantities. And other writers who have made the like attempt, have fallen into ſimilar abſurdities; amongſt whom we may reckon the celebrated Mr. Leonhard Euler, in his laſt work, called *The Elements of Algebra*, in two large volumes, octavo, which has been commended by ſome perſons as a very eaſy and clear treatiſe on that ſcience. But, upon examining it, I have found the firſt volume of it to abound with errors and difficulties ariſing from a moſt compleat adoption and very frequent uſe of this perplexing and abſurd doctrine of negative quantities, for which he ſeems to have an uncommon attachment.

After ſeeing that ſo able a mathematician as Monſieur *Clairaut* could be ſo far blinded and puzzled by this ſtrange doctrine of negative quantities (unneceſſarily introduced into the otherwiſe clear and ſimple ſcience of Algebra, or Univerſal Arithmetick,) as to reaſon ſo very weakly in ſupport of it as we have ſeen he did in the foregoing paſſage, it is ſurely high time for every true lover of this ſcience, who is zealous for the honour of it's purity and perſpicuity, to exclaim as the good Archbiſhop *Tillotſon* did with reſpect to the Athanaſian Creed, " *I wiſh we were fairly rid of it!*"

FINIS.

PRINCIPLES

OF

A L G E B R A.

THE

PRINCIPLES

OF

A L G E B R A:

OR

THE TRUE THEORY OF EQUATIONS

ESTABLISHED ON

MATHEMATICAL DEMONSTRATION.

PART THE SECOND.

BY

WILLIAM FREND.

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1799.

TO THE

REV. ROBERT TYRWHITT, M. A.

THIS VOLUME,

INDEBTED TO HIM, IN GREAT MEASURE,

FOR ITS EXISTENCE,

IS DEDICATED

BY THE AUTHOUR.

PREFACE.

FROM the age of Vieta, the father, to this of Maſeres, the reſtorer of Algebra, many men of the greateſt abilities have employed themſelves in the purſuit of an idle hypotheſis, and have laid down rules not founded in truth, nor of any ſort of uſe in a ſcience admitting in every ſtep the plaineſt principles of reaſoning. If the name of Sir Iſaac Newton appears in this liſt, the number of the advocates for errour muſt be conſiderable. It is, however, to be recollected, that for a much longer period, men ſcarcely inferiour to Newton in genius, and his equals probably in induſtry, maintained a variety of poſitions in philoſophy, which were overthrown by a more accurate inveſtigation of nature; and, if the name of Ptolemy can no longer ſupport his epicycles, nor that of Des Cartes his vortices, Newton's dereliction of the principles of reaſoning cannot eſtabliſh the fallacious notion, that every equation has as many roots as it has dimenſions.

This

This notion of Newton and others is founded on precipitation. Inſtead of a patient examination of the ſubject, an hypotheſis, which accounts for many appearances, is formed: where it fails unintelligible terms are uſed: in theſe terms indolence acquieſces: much time is waſted on a jargon which has the appearance of ſcience, and real knowledge is retarded. Thus volumes upon volumes have been written on the ſtupid dreams of Athanaſius, and on the impoſſible roots of an equation of n dimenſions.

It was very early perceived that equations in certain forms admitted ſeveral roots, and that by multiplying together complex terms in certain forms, terms ſimilar to theſe equations might be produced. Hence, from the natural deſire of mankind to come as ſoon as poſſible to a general concluſion, the thought ſuggeſted itſelf, that all equations might be produced by a multiplication of ſimilar complex terms, and that the analogy which had been obſerved between the coparts and roots of certain equations might be extended in general to all equations. The idea thus advanced was without ſufficient attention to the natural difference, which ſubſiſts in the different order of equations, adopted. It is true, that all the equations which have more than one term may be formed by the multiplication of two or more complex terms; but it depends

depends upon the form of an equation, whether it admits of a multiplication by a number of complex terms in the forms propoſed equal to the higheſt index of the given equation. Thus let

$$x^2 + a\,x = k.$$

Let the root of the above equation be equal to b, then $\overline{x - b} \times \overline{x + a + b}$ is equal to $x^2 + a\,x - k$; but a and b are added to x and conſequently $a + b$ cannot be a root of the given equation. In the ſame manner let

$$x^m + a\,x^n + b\,x^o + c\,x^p + d\,x^q \ldots\ldots = k.$$

If l is one root, then $x - l$ multiplied into ſome complex term may form the equation; but this complex term is equal to an aſſignable number, and the number ſubſtituted for x in this complex term, which makes it equal to the aſſignable number, cannot be the root of the propoſed equation. With true reaſoners this would have been ſufficient to make them reject all thoughts of producing every equation by the multiplication of a number of complex terms equal to the higheſt index in the given equation; but unfortunately a fiction was introduced, and ſcience was placed upon a level with the chicanery of courts of law in ſome countries, where the firſt ſtep to obtain redreſs for certain injuries commences with a falſehood. Though it is certain that in the inſtance above, where $\overline{x + a} \times \overline{x + a + b}$ produces the given

equation, $a + b$ is added to x, yet by a ſtrange fiction it was to be ſuppoſed that $a + b$ is taken from x, and that $a + b$ is equal to a number c, to be called either a negative or an impoſſible number. The nature of theſe fictitious numbers now became an object of inquiry, and, inſtead of ſearching after the roots of an equation, even grave men, imitating the philoſophers of a well-known region, who were extracting ſun-beams from cucumbers, waſted the midnight oil juſt as profitably in ſettling the rights and privileges of impoſſible quantities.

The fruits of their lucubrations I ſucked in with the firſt milk of alma mater; and, if the good old lady had not driven me from my books and from ſtudies of a much ſuperiour nature, it is not improbable that I might ſtill have looked at theſe ſubjects through her ſpectacles. It is of little conſequence to the publick how, or why, I came back to my former ſtudies; ſuffice it, that being no longer of an age nor of a diſpoſition to take for granted an old wive's tale, I ſtarted at the poſtulate to make a load of corn out of a buſhel. So ſtrange a demand now excited my inquiry into the principles of a ſcience whoſe objects are all of a determinate nature, and after conſiderable reſearches I diſcovered the direct demonſtrations which this volume contains. A ſpecimen of them I gave

laſt

laſt ſummer in a Letter * to the Vice Chancellor of the Univerſity of Cambridge on the vacancy of a profeſſerſhip, which was filled up by one whoſe pretenſions to be a candidate were, in the opinion of moſt people, againſt the ſpirit, and, in that of many, againſt the letter of the ſtatute. In my opinion his election was againſt both; but, if he does any thing worthy of that profeſſerſhip, I ſhall not be very ſcrupulous about his right of obtaining it, and ſhall not, moſt aſſuredly, in theſe times, diſturb him in the poſſeſſion of it.

In this work the number of roots in an equation is determined, not by a fiction, but on certain and undeniable principles; a direct demonſtration is given of the relation of the coparts and roots in ſeveral equations, and the mode pointed out for all equations; the limiting equations are diſcovered by a ſimple principle which appears to me capable of great extenſion. In this and ſimilar queſtions fluxions are generally applied, that is, the new and unneceſſary quality of velocity is introduced in queſtions depending on mere addition and ſubtraction. My principle is ſimply this. Let a and b be certain numbers,

* A Letter to the Vice Chancellor of the Univerſity of Cambridge, by William Frend, Candidate for the Lucaſian Profeſſerſhip. Price 6d. White, Fleet-ſtreet; and Flower and Deighton, Cambridge.

z and y variable numbers; then if

$$a = b + Az - By + Cz^2 - Dy^2 + Ez^3 - Fy^3 \ldots$$

the difference between the variable terms added and thoſe ſubtracted muſt be equal either to a given number or to nothing, otherwiſe a would be a variable number. Now the difference cannot in my demonſtrations be in any caſe a given number, for both z and y may be taken leſs than any aſſignable number. Conſequently, if this difference was any determinate number, when z and y were each equal to ſome given numbers, this difference, when z and y were both made leſs than any aſſignable number, would be leſs than the preceding determined number, that is, the difference would be a variable number which might be expreſſed by v, and then a would be equal to $b \pm v$, which is impoſſible, for a can be equal to only one number. The advantage of this demonſtration is, that z and y may be tried in any given equation, and the reſult made evident to the learner in every inſtance. If $12x - x^2 = 20$, then the limiting equation $12 = 2x$ or $x = 6$, and if z is made equal to unity then y is equal to unity, and 6 ± 1 being ſubſtituted for x in the given equation produce the ſame reſults; if $z = 2$, then $y = 2$, and ſo on *.

* See Pages 26 — 29, 67 — 69.

I have alſo given in this work a conſiderable number of examples, particularly from Raphſon; for I have found by experience that a learner, who is ſatisfied with general expreſſions and general demonſtrations without trying the reſult in practice ſeldom acquires any ſolid knowledge, and accordingly it is a great point with me in my preſent mode of teaching to accompany in every inſtant the theory with practice. The methods adopted by ſeveral eminent mathematicians in the ſolution of equations are pointed out with their merits and defects: and here I ought particularly to obſerve that the demonſtration given in pages 13 and 14 is taken chiefly from Dr. Hutton's eſſays; and, though I cannot attribute to him the whole merit of that method, I can eaſily conceive that he did not derive any aſſiſtance from others; and his great merits will not ſuffer by the juſt tribute of applauſe being beſtowed on De Lagny his predeceſſer in this diſcovery. By the compariſon of my mode of dividers with that of Dr. Hutton's by double poſition, pages 96, 97, I do not by any means conſider the queſtion as thoroughly decided, for though I have no doubt in my own mind on the preference of my mode to his in all the forms of equations, which are the ſubject of this work, I muſt leave to the learner and to farther inquiry on my own part thoſe equations, where the unknown term is under

der a radical band, and where at preſent I conjecture only that my mode has the advantage.

It might be proper here to acknowledge again the egregious overſight committed in the firſt part of my Algebra, page 213, if I had not already done it in the Monthly Magazine, and to my miſtake the publick is indebted for an appendix, in which Baron Maſeres has given a complete examination of Cardan's rule, and a moſt accurate compariſon of Ferrari's ſolution, with that of Raphſon, by his method of approach. I may preſume alſo to hope that the tutours of the Univerſity of Cambridge will recommend to their pupils the five laſt pages of that appendix to rectify their notions on the fiction of negative and impoſſible quantities; and, if they dilate on the concluding paragraph, they will probably make an intereſting compariſon between the ſimilar fictions of modern algebraiſts and antient divines.

WILLIAM FREND.

N° 9, Inner Temple-lane.

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PRINCIPLES OF ALGEBRA.

PART THE SECOND.

CHAP. I.

ON EQUATIONS IN GENERAL.

FROM the former part of this work it appears, that ſome equations have more roots than one: and as the diſcovery of the roots of equations is a principal part of Algebra, a perſon cannot be a proficient in the ſcience, unleſs he ſhall be able, on certain grounds, to determine how many roots an equation in any form propoſed to him can admit. The Appendix has alſo ſhewn with what difficulty the ſolution of ſeveral equations muſt be attended; and, as this difficulty increaſes with the number of terms on the unknown ſide of an equation, it will be of uſe to examine the means of ſhortening this labour, and of confining the firſt approaches to the roots within ſome aſſignable limits. Hence this part of the principles of Algebra will be confined chiefly to theſe two enquiries; into

the number of roots in any equation, and into the limits within which theſe roots are confined; and though the forms of equations are more numerous than the leaves of autumn in Vallombroſa; and out of the myriads of theſe forms, the ingenuity of man has not hitherto diſcovered a general expreſſion for the root in one form out of a million; yet, by certain methods, the root of any equation which can be expreſſed by decimal arithmetick, will, by a degree of labour proportional to the number of figures in the root and the number of terms in the equation, be diſcovered.

As the forms of equations are ſo numerous, it will be expedient to adopt ſome mode of dividing them into claſſes: and theſe claſſes ſhould be, if poſſible, of ſuch a nature, that a general property may go through the forms of every claſs. There is, firſt, a general diviſion of equations according to the nature of their unknown terms; for they may admit either of only one unknown number with its powers, or of ſeveral unknown numbers with their powers; or the index of the number in any term may be an unknown number. Thus the equation:

$x^n \pm ax^{n-1} \mp bx^{n-2} \pm cx^{n-3} \ldots\ldots = k$ has on one ſide only x and its powers.

The equation

$x^n \pm vx^{n-1} \mp vyx^{n-2} \pm vyzx^{n-3} \mp \ldots\ldots\ldots = k$

has ſeveral unknown numbers v, y, z, &c. with their powers.

The

The equation

$x^{x} \pm b a^{x-1} y + c a^{x-2} \pm d a^{x-3} \ldots\ldots\ldots = k$ has not only ſimple unknown numbers with their powers; but the indexes alſo of ſeveral numbers are unknown.

Of theſe equations, the firſt only will make the principal part of this enquiry; namely, thoſe forms in which a ſimple unknown number or its powers are involved; and equations of this kind will be claſſed according to the number of terms on the unknown ſide. Thus all equations will be conſidered of the firſt claſs, which have only one unknown term, and conſequently only one root; and theſe equations come under the form $x^{n} = k$.

Equations of the ſecond claſs have only two unknown terms, and cannot have more than two roots, and are reducible to one or the other of the three forms:

$$x^{m+n} + a x^{n} = k.$$
$$x^{m+n} - a x^{n} = k.$$
$$a x^{n} - x^{m+n} = k.$$

Equations of the third claſs have three unknown terms; of the fourth claſs, four unknown terms, and ſo of the higher claſſes. One general rule pervades all equations: namely, that no equation in any claſs can have more roots than it has unknown terms; and in the forms of each claſs the number of roots depends partly upon the coparts of the unknown terms, and partly upon the

changes of the marks of adding or taking away. In equations whoſe unknown terms are ranged according to the order of their powers, there can be only one root, if there is only one or no change of the marks; and in equations, not ranged according to the order of the powers, there will be only one root, if there is a certain relation between the known term of the coparts of the unknown terms. Thus equations in theſe forms

$$x^n + a x^{n-1} + b x^{n-2} + \ldots + \ldots + \ldots + = k$$

$$x^n - a x^{n-1} - b x^{n-2} - \ldots - \ldots - \ldots - = k,$$

can have only one root.

CHAP. II.

EQUATIONS OF THE FIRST CLASS.

Equations of the firſt claſs, after bringing the known terms to one ſide, and the unknown to the other, and freeing the unknown term of its known parts, are of this form:

$$x^n = k.$$

To find therefore the number x, the n^{th} root of k is to be found.

If $n = 2$, the ſecond root may be found by the rules laid down in the firſt Part (page 92, and the following pages).

If

If $n = 3$, the third root may be found by the rules laid down in the firſt Part, (page 96, and the following).

If $n = 4$, the fourth root may be found upon ſimilar principles, and ſo on for any number n.

But theſe methods of finding the root are very troubleſome, eſpecially in the higher powers; and therefore, when the root is contained within ſix or ſeven figures, it is diſcovered in the eaſieſt manner by the logarithmical tables; or, if a number is required which ſhall approach as nearly as poſſible to the root of a given number, the firſt ſix or ſeven figures are diſcovered in the eaſieſt manner by logarithms. Thus, let

$$x^2 = 2.$$

$$\therefore 2, \text{ Log. } x = \text{Log. } 2 = ,30103$$

$$\therefore \text{ Log. } x = ,150515$$

$$\text{Log. } 1,41421 = ,1505139.$$

Diff. between Log. 1,414210 and Log. 1,414220 = ,0000031.

$$\frac{\text{Diff.}}{10} = ,00000031.$$

$$\therefore \frac{3.\text{Diff.}}{10} = ,00000093.$$

$$\therefore \text{Log. } 1,414213 = \left.\begin{matrix} ,1505139 + \\ + ,00000093 \end{matrix}\right\} = ,15051483.$$

$$\text{Log. } 1,4142130 = ,15051483$$

$$\text{Log. } 1,4142140 = ,15051514.$$

$$\therefore \text{Diff.} = ,00000031.$$

$\therefore$

$$\therefore \frac{\text{Diff.}}{10} = ,00000031$$

$$\therefore \frac{5 \text{ Diff.}}{10} = ,000000155.$$

Add Log. 1,4142130 = ,15051483

$\therefore$ Log. 1,4142135 = ,150514985.

Log. 1,4142135 = ,150514985

Log. 1,4142136 = ,150515016

$\therefore$ Diff. = ,000000031

$$\frac{\text{Diff.}}{10} = ,00000000031$$

$$\therefore \frac{6. \text{ Diff.}}{10} = ,000000000186$$

Add Log. 1,4142135 = ,150514985.

$\therefore$ Log. 1,41421356 = ,1505150036;

but ,1505150036 is greater than ,150515; therefore 1,41421356 is by the tables greater than the second root of two, which is not true, for the second root of two is greater than 1,41421356, and the error is owing to the defect in the tables, which are calculated only to seven places of decimals. But the figures 1,4142135 are right.

Let $x^3 = 37945$.

$\therefore$ 3 Log. x = Log. 37945 = 4,5791546

$\therefore$ Log. x = 1,526384866 ....

$\therefore$ x = 33,603522 ....

The firſt ſeven figures are true.

Let

Let $x^4 = 2741583974$.

Suppoſe $x = 10y$.

$\therefore 10^4 y^4 = 2741583974$

$\therefore y^4 = 274158,3974$

$\therefore 4 \text{ Log. } y = \text{Log. } 274158,3 \text{ nearly} = 5,43800168$

$\therefore \text{Log. } y = 1,35950042$

$\therefore y = 22,882330\ldots$

$\therefore x = 228,82330\ldots,$

The ſeven firſt figures are true.

Let $x^5 = 2327834559873$.

Suppoſe $100y = x$

$\therefore \overline{100}^5 y^5 = 2327834559873$

$\therefore y^5 = 232,7834559873$

$\therefore 5 \text{ Log. } y = \text{Log. } 232,78345 \text{ nearly, } = 2,366952119$

$\therefore \text{Log. } y = ,473390422$

$\therefore y = 2,9743381 \ldots.$

$\therefore x = 297,43381 \ldots.$

The ſeven firſt figures are true.

If tables of Logarithms are not at hand, recourſe may be had to the mode of diſcovering the root by gradual approaches to it; for the firſt figures being found, the unknown number may be made equal to the ſum or difference of the known number and another unknown number.

Let $x^n = k$.

And ſuppoſe x to lie between the two numbers c and d.

Then ſuppoſe $x = c - z$ or $d + z$

$\therefore \overline{c - z}^n = k = \overline{d + z}^n$

$\therefore$

$$\therefore c^n - n c^{n-1} z + n.\frac{n-1}{2}. c^{n-2} z^2 \ldots\ldots = k$$

$$\text{or } d^n + n d^{n-1} z + n.\frac{n-1}{2} d^{n-2} z^2 \ldots\ldots = k$$

Neglect all the powers of z.

$$\therefore c^n - n c^{n-1} z = k \quad \therefore z = \frac{c}{n} - \frac{k}{n c^{n-1}} = \frac{1}{n}.\overline{c - \frac{k}{c^{n-1}}}$$

$$d^n + n d^{n-1} z = k \quad \therefore z = \frac{k}{n d^{n-1}} - \frac{d}{n} = \frac{1}{n}.\overline{\frac{k}{d^{n-1}} - d}$$

Hence, by taking away z thus found from c by the firſt expreſſion, or by adding it to d as found by the ſecond expreſſion, a number nearer to x in the given equation than either c or d is found. Let theſe numbers be e or f, and, by proceeding as before, a number ſtill nearer to x is found, and ſo on for ever.

Let $x^2 = 2$.

If $x = 1$, $x^2 = 1$; and if $x = 2$, $x^2 = 4$. Therefore x is between one and two

Let $x = 1 + z$.

$$\therefore 1 + 2z + z^2 = 2$$

neglecting z^2

$$z = \frac{2-1}{2} = \frac{1}{2} = ,5 \text{ which is too great for } z = ,5 - \frac{z^2}{2}$$

$$\therefore x > 1,5.$$

Let $x = 1,5 - z$.

$$\therefore \overline{1,5}^2 - 2.\ 1,5\, z + z^2 = 2;$$

neglecting

neglecting z^2,

$$\overline{1,5}^2 - 2 = 3,0\, z.$$

$$\therefore z = ,5 \times 1,5 - \tfrac{2}{3} = \left\{ \begin{matrix} ,75 - \\ -,666 \end{matrix} \right\} = ,083$$

which is too small.

$$\therefore x > 1,5 - ,083$$
$$\therefore x > 1,4166.$$

Since x is less than 1,4166, it may be either greater or less than 1,416. Therefore make $x = 1,416 \pm z$;

$$\therefore \overline{1,416}^2 \pm 2 . 1,416 . z + z^2 = 2 .$$

$\therefore$ neglecting z^2,

$$\overline{1,416}^2 \mp 2 = 2 . 1,416\, z$$

$$\therefore z = \frac{1,416}{2} \mp \frac{1}{1,416}$$

$$= ,708 \mp ,706214689$$

$$= ,001785311 \text{ too small}$$

$$\therefore x = 1,414214688 \ldots$$

Make $x = 1,414214$, and proceed as before.

Let $x^3 = 37945$.

Suppose $x = 10\, y$.

$$\therefore 10\, y^3 = 37945$$

$$\therefore y^3 = 37,945$$

$\therefore y$ is between 3 and 4, but nearest to 3.

Let it therefore be equal to $3 + z$.

$$\therefore 3^3 + 3 \cdot 3^2 z + 3 \cdot 3\, z^2 + z^3 = 37,945,$$

neglecting z^2 and z^3;

$$3 \cdot 3^2 z = 37,945 - 3^3$$

$$\therefore z = \frac{37,945}{9 \cdot 3} - 1$$

$$= \frac{4,2161}{3} - 1$$

$$= 1,40537 - 1 = ,40537 \text{ too great.}$$

$$\therefore y > 3,40537.$$

Let $y = 3,4 - z.$

$$\therefore \overline{3,4}^3 - 3 \cdot \overline{3,4}^2 z + \text{---} = 37,945$$

$$\therefore \overline{3,4}^3 - 37,945 = 3 \cdot \overline{3,4}^2 \cdot z$$

$$\therefore \frac{3,4}{3} - \frac{37,945}{3 \cdot \overline{3,4}^2} = z = \begin{matrix} 1,133333 \\ -\ 1,09414 \end{matrix}$$

$$= ,039818 \text{ too small.}$$

$$\therefore y > 3,36081.$$

Make $y = 3,36 \pm z.$

From theſe inſtances it appears, that by every operation a nearer approach is made to the required root; and in ſuch laborious work it is uſeful to adopt every method for ſhortening the number of operations. Now this will be done by a ſlight attention to the neglected powers of z.

Thus, in the inſtance $x^2 = 2$, z is found to be $,5 - \frac{z^2}{2}$. Conſequently z cannot be equal to ,5; nor $\frac{z^2}{2}$ to $\frac{,25}{2}$, or ,125. Therefore if ,125 is taken from ,5 the reſult ,375 will give a number leſs than z. Conſequently z is between the numbers ,5 and ,375; and it is nearer to ,375 than to ,5. Hence, inſtead of uſing $1,5 - z$ for the next approach, it would be expedient and juſtifiable to try $1,4 \pm z$ for x.

$$\therefore \overline{1,4}^{\,2} \pm 2 \cdot 1,4 \cdot z + z^2 = 2$$

$$\therefore 1,4 \pm 2 \cdot z + \frac{z^2}{1,4} = \frac{2}{1,4} = \frac{1}{,7} = 1,428571.$$

From this equation it appears z is to be added to 1,4; and consequently the equation becomes

$$2z = 1,428571 - 1,4 - \frac{z^2}{1,4}$$

$$\therefore z = \frac{,02857142}{2} \ldots - \frac{z^2}{2.1,4}$$

$$\therefore z = ,0142857 \ldots - \frac{z^2}{2.1,4}.$$

Hence z is leſs than ,0142857, ... and z^2 is nearly equal to $\overline{,014}^{\,2}$. Therefore $\frac{z^2}{2.1,4} = \frac{\overline{,014}^{\,2}}{2.1,4}$ nearly $= \frac{,001 \times ,014}{2 \times 0,1} = \frac{,000014}{2.0,1} = ,00007$ nearly.

$$\therefore z = \left\{ \begin{matrix} ,0142857 \\ -,00007 \end{matrix} \right\} = ,0142157 \text{ too great.}$$

$$\therefore x = 1,414215 \pm z.$$

By proceeding in this manner another approach will be made, which will double the number of figures.

In the inſtance $y^3 = 37,945$ the equation derived was $3^3 + 3 \cdot 3^2 z + 3 \cdot 3z^2 + z^3 = 37,945$ (ſee page 9)

$$\therefore 3.z = \frac{37,945}{9} - 3 - z^2 - \frac{z^3}{9}$$

$\therefore$

$$\therefore 3z = \left\{ \begin{matrix} 4,2161 \ldots - z^2 - \frac{z^3}{9} \\ -3, \end{matrix} \right\} = 1,2161 \ldots\ldots$$

$$- z^2 - \frac{z^3}{9}.$$

$$\therefore z = ,40537037037 - \frac{z^2}{3} - \frac{z^3}{3 \cdot 9}$$

$$\therefore \frac{z^2}{3} = \frac{,16}{3} \text{ nearly} = ,0533$$

$$\frac{z^3}{3 \cdot 9} = \frac{,064}{3 \cdot 9} = ,00237 \ldots$$

$$\therefore z = \left\{ \begin{matrix} ,40537037 \\ - ,0557 \end{matrix} \right\} = ,3496 \ldots\ldots$$

$$\therefore y = 3,35 \pm z$$

$$\therefore \overline{3,35}|^3 \pm 3 \cdot \overline{3,35}|^2 z + 3 \cdot 3,35 z^2 \pm z^3 = 37,945$$

$$\therefore 3,35 \pm 3z + \frac{3 \cdot z^2}{3,35} \pm \frac{z^3}{\overline{3,35}|^2} = \frac{37,945}{\overline{3,35}|^2} = \frac{7,589}{,67 \times 3,35}$$

$$= \frac{1,5178}{,67 \times ,67} = 3,381153 \ldots$$

$$\therefore 3z = 0,0311538 - \frac{3z^2}{3,35} - \frac{z^3}{\overline{3,35}|^2}$$

$$\therefore z = 0,0103846 - \frac{z^2}{3,35} - \frac{z^3}{\overline{3,35}|^2}$$

$$\therefore \frac{z^2}{3,35} = \frac{\overline{,0103}|^2}{3,35} = ,000031669$$

$$\therefore z = \left\{ \begin{matrix} 0,0103846 \\ - ,00003166 \end{matrix} \right\} = ,01035297 \text{ nearly.}$$

$$\therefore y = \left\{ \begin{matrix} 3,35 \\ + ,0103529 \end{matrix} \right\} = 3,3603529 \text{ nearly.}$$

$$\therefore x = 33,60352 \pm z.$$

Hence

Hence it appears, that by taking two or three of the figures firſt found for z, and thence obtaining a number nearly equal to z^2, a correction may be made to z firſt found; and thus a greater number of true figures will be found in each operation, and conſequently the labour upon the whole will be very much ſhortened.

As the great object in all the operations is to make $\frac{\frac{n-1}{2} z^2}{a}$ bear a very ſmall proportion to z; a general expreſſion for the firſt near number to the root may, when logarithmical tables are not at hand, be applied with great ſucceſs.

Let $x^n = k$ and a a number nearly equal to x.

$$\therefore \overline{a + z}|^n = k$$

$$\therefore a^n + n\,a^{n-1} z + n.\frac{n-1}{2}\, a^{n-2} z^2 + \ldots\ldots = k$$

$$\therefore n\,a^{n-1} z + n.\frac{n-1}{2}\, a^{n-2} z^2 + \ldots\ldots = k - a^n$$

$$\therefore z + \frac{n-1}{2}\frac{z^2}{a} + \frac{n-1}{2}\cdot\frac{n-2}{3}\frac{z^3}{a^2} + \ldots\ldots\ldots$$

$$= \frac{k-a^n}{n\,a^{n-1}} = \frac{k-a^n}{n\,a^n} \times a$$

$\therefore$ by neglecting z^2 and the higher powers,

$$z = \frac{k-a^n}{n\,a^n} \times a \text{ nearly.}$$

$\therefore$

$$\therefore a + z = \frac{k - a^n}{n\,a^n} \times a + a.$$

This expreſſion $\frac{k - a^n}{n\,a^n} \times a + a$ is greater than x.

Since $k = a^n + n\,a^{n-1}z + n \cdot \frac{n-1}{2} a^{n-2} z^2 + \ldots$

$\ldots\ldots = A.$

Multiply both ſides by $\overline{n-1}$; and add a^n to each ſide.

$$\therefore \overline{n-1} \cdot k + a^n = n\,a^n + n \cdot \overline{n-1} \cdot a^{n-1} z +$$

$$\overline{n-1} \cdot n \cdot \frac{n-1}{2} a^{n-2} z^2 + \ldots = B.$$

Alſo multiply the former equation A, both ſides by a, and ſubtract a^{n+1}

$$\therefore \overline{k - a^n} \times a = n\,a^n z + n \cdot \frac{n-1}{2} a^{n-1} z^2 + \ldots = C$$

$$\therefore \frac{C}{B} = \frac{k - a^n}{\overline{n-1} \cdot k + a^n} \times a = \frac{n\,a^n z + n \cdot \frac{n-1}{2} a^{n-1} z^2 + \ldots}{n\,a^n + n \cdot \overline{n-1} \cdot a^{n-1} z \ldots}$$

$$= \frac{n\,a^n z}{n\,a^n} \text{ nearly} = z$$

$$\therefore a + z = a + \frac{k - a^n}{\overline{n-1} \cdot k + a^n} \times a \text{ nearly.}$$

Hence, ſince $\frac{k - a^n}{n\,a^n} \times a + a$ is greater than x, and $a + \frac{k - a^n}{\overline{n-1} \cdot k + a^n}$ is leſs than x, by adding together the two upper parts of theſe fractions for a new upper part, and the

the lower parts for a new lower part, a fraction will be found, which added to a is nearer to x than either of the other two terms*:

$$\overline{k - a^n} \times a + k - a^n \times a = \overline{k - a^n} \times 2a = D$$

$$n a^n + \overline{n - 1} \cdot k + a^n = \overline{n + 1} \cdot a^n + \overline{n - 1}\, k = E$$

$$\therefore a + \frac{D}{E} = a + \frac{k - a^n}{\overline{n + 1} \cdot a^n + \overline{n - 1} \cdot k} \times 2a. = x \text{ nearly}\dagger.$$

Let

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* If either of two fractions is nearly equal to a given number, the sum of their upper parts, divided by the sum of their lower parts, is still nearer to equality with that number than either of the two fractions.

$$\text{Let } \frac{a}{b} = m + x;$$

$$\frac{c}{d} = m - y;$$

$$\text{Then } a = b\,m + b\,x$$

$$\text{And } c = d\,m - d\,y$$

$$\therefore a + c = b\,m + d\,m + b\,x - d\,y$$

$$\therefore \frac{a + c}{b + d} = m + \frac{b\,x}{b + d} - \frac{d\,y}{b + d}.$$

But $\frac{b\,x}{b+d}$ is less than x; and $\frac{d\,y}{b+d}$ is less than y. Therefore $m + \frac{b\,x}{b+d} - \frac{d\,y}{b+d}$ is less than $m + x$; and consequently $\frac{a+c}{b+d}$ is nearer to m than $\frac{a}{b}$ is to m. And $m + \frac{b\,x}{b+d} - \frac{d\,y}{b+d}$ is greater than $m - y$. Therefore $\frac{a+c}{b+d}$ is nearer to m than $\frac{c}{d}$ is to m.

† This is De Lagny's expreſſion. Dr. Hutton ſimplifies this expreſſion in appearance by bringing it to one common lower part:

$a +$

Let $x^2 = 2$.

$\therefore a = 1$; $k = 2$; $n = 2$.

$$\therefore a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = 1 + \frac{2-1}{2+1 \cdot 1+1 \cdot 2}$$

$$\times 2 = 1 + \frac{1}{5} \times 2 = 1 + \frac{2}{5} = 1{,}4.$$

Again let $a = 1{,}4$.

$$\therefore \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = \frac{2 - \overline{1{,}4}^2}{3 \cdot \overline{1{,}4}^2 + 2} \times 2 \cdot 1{,}4$$

$$= \frac{2 - \overline{1{,}4}^2}{3 \cdot {,}7 \times 1{,}4 + 1} \times 1{,}4 = \frac{1 - {,}7 \times 1{,}4}{3 \times {,}7 \times {,}7 + {,}5} \times 1{,}4 =$$

$$\frac{1 - {,}98}{1{,}47 + {,}5} \times 1{,}4 = \frac{{,}02 \times 1{,}4}{1{,}97} = \frac{{,}028}{1{,}97} = {,}0142319.$$

$$\therefore a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = 1{,}41421319 \ldots$$

$$\ldots\ldots = x.$$

Of theſe the firſt ſeven figures are true; and for the next approach a might be made equal to 1,4142132: but the raiſing of ſo great a number to a ſecond power is very laborious; and as the divider alſo conſiſts of a great number of figures, though the reſult will give a greater number of figures than by the expreſſion $x = \frac{k}{2a} - \frac{a}{2} - \frac{x^2}{2a}$, yet

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$a + \frac{k - a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2a = \frac{\overline{n+1} \cdot a^{n+1} + \overline{n-1} \cdot ak + 2ak - 2a^{n+1}}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k}$

$= \frac{\overline{n+1} \cdot k + \overline{n-1} \cdot a^n}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times a$; but he increaſes the trouble of the operation.

this

this latter expreſſion will be found far more convenient in practice.

Let $x^3 = 37{,}945$.

$$\therefore a = 3 \quad k = 37{,}945 \quad n = 3$$

$$\therefore \frac{k - a^3}{\overline{n+1} \cdot a^n + \overline{n-1} \cdot k} \times 2{,}a = \frac{37{,}945 - 27}{4 \cdot 27 + 2 \cdot 37{,}945} \times 6 =$$

$$\frac{10{,}945 \times 3}{2 \times 27 + 37{,}945} = \frac{2{,}189 \times 3}{{,}4 \times 27 + 7{,}589} = \frac{2{,}189 \times 3}{18{,}389}$$

$$= \frac{{,}312714285\overline{71} \times 3}{2{,}627} = {,}119038 \times 3 = {,}3571$$

$$\therefore x = 3{,}3571.$$

Now let $a = 3{,}35$,

and the learner by purſuing a ſimilar operation may compare the effects of it with De Lagny's method.

The mode of dividers may alſo be applied to equations in this claſs, for the diſcovery of a few of the firſt figures. Thus, let $x^2 = 2$; therefore if

x	= 1	2	1,5	1,4	1,414	1,415
$\frac{2}{x}$	= 2	1	1,33	1,428	1,414422	1,413427
Diff.	=		,1β	,028	,000422	,001572.

Hence, x is between 1,5 and 1,40 but nearer to 1,4, and the diff. ,16 + ,028 = ,188.*

∴

* The ſum of the differences in the third line may be divided into ten, twenty, thirty, &c. equal parts, according as their correſponding numbers in the upper line differ by ,1 ,2 ,3, &c. and as often as one of theſe parts is contained in either of the lower differences, ſo many parts of ,1

$$\therefore \frac{\text{Diff.}}{10} = ,0188$$

$$\text{and } 8 \times ,0188 = ,1504 > ,16$$

$$\therefore 1,5 - ,08 \ldots = x = 1,41 \ldots .$$

$$\text{and } ,0188 \times 1 > ,028$$

$$\therefore 1,4 + ,01 \ldots . = x = 1,41 \ldots .$$

$$\text{Alſo } \frac{,16}{,0188} = 8,51$$

$$\therefore 1,5 - ,0851 . = 1,4148 \ldots = x \text{ nearly.}$$

$$\frac{,028}{,0188} = 1,489$$

$$\therefore 1,4 + ,01489 = 1,414 \ldots = x \text{ nearly.}$$

,2 ,3, &c. may be added to or taken from the correſponding upper numbers, and the reſult will be a number much nearer to the root required. Or, ſince the root lies between two numbers in the upper line, and the differences in the lower line are very nearly in the proportion of the exceſs of one number in the upper line above the root to the defect of the other number from the root, the following proportion will be nearly true:

$$,16 : ,028 :: ,1 - z : z$$

$$\therefore ,16\,z = ,0028 - ,028\,z$$

$$\therefore ,188\,z = ,0028$$

$$\therefore z = \frac{,0028}{,188} = ,01489$$

$$\therefore x = 14 + ,01489 = 14,1489 \text{ nearly.}$$

Again:

$$,028 : ,16 :: ,1 - z : z$$

$$\therefore ,028\,z = ,016 - ,16\,z$$

$$\therefore ,188\,z = ,016$$

$$z = \frac{,016}{,188} = ,0851$$

$$\therefore x = 1,5 - ,0851 = 1,4148 \ldots \text{ nearly.}$$

For

For the firſt approach therefore x ſhould be made $1{,}414 + z$

Let $x^3 = 37{,}945$.

$$x = 3 \quad 4 \quad 3{,}3$$
$$\frac{37{,}945}{x} = 12{,}648 \quad 9{,}4 \quad 11{,}49$$
$$x^2 = 9 \quad 16 \quad 10{,}89$$
$$\text{Diff.} = 3{,}648 \quad 0{,}60$$
$$\frac{3{,}648 + 0{,}60}{30} = \frac{4{,}248}{30} = 1{,}416$$
$${,}4 \times 1{,}416 = {,}5764 > {,}60$$
$$\therefore x = 3{,}3 + {,}04 \ldots = 3{,}34 \ldots$$

For the firſt approach therefore make $x = 3{,}34 + z$.

Let $y^5 = 232{,}7834559873 = k$.

$$y = 2 \quad 3$$
$$\frac{k}{y} = 116 \quad 77$$
$$y^4 = 16 \quad 81.$$

Here y is very nearly equal to 3, and it will be better to make $y = 3 - z$ than to attempt further approaches by dividers, as the next trial muſt be made with 2,9 or 2,96.

CHAP. III.

EQUATIONS OF THE SECOND CLASS.

Equations of the ſecond claſs have two unknown terms, and are all reducible to one or other of theſe forms.

$$1.\ x^{m+n} + a\,x^n = k$$
$$2.\ x^{m+n} - a\,x^n = k$$
$$3.\ a\,x^n - x^{m+n} = k$$

The two firſt forms admit only of one root; the third form admits of two roots. The firſt form admits only of one root: for $x^{m+n} + a\,x^n = x^n \times \overline{x^m + a}$, and both parts x^n, and $x^m + a$ increaſe or decreaſe together; and conſequently the product increaſes or decreaſes with the increaſe or decreaſe of x: and, if this product is in any inſtance equal to k, by increaſing or decreaſing x the product muſt be made greater or leſs than k.

In the ſame manner $x^{m+n} - a\,x^n$ in the ſecond form is equal to $x^n \times \overline{x^m - a}$, whoſe parts alſo both increaſe or decreaſe together, and conſequently there can be but one root to an equation in this form.

But $a\,x^n - x^{m+n}$ in the third form is equal to $x^n \times \overline{a - x^m}$, whoſe parts do not increaſe and decreaſe together: for, if x^n increaſes, $a - x^m$ decreaſes; and, if x^n decreaſes, $a - x^m$ increaſes. Conſequently, if this product is equal to k, when $x = d$, by increaſing x the product may firſt increaſe and then decreaſe, and conſequently

quently another number c greater than d may make $x^n \times \overline{a - x^m} = k$. But this form does not admit of more than two roots. For the product increaſes by addition to x, from the leaſt aſſignable number till it becomes the greateſt poſſible, and then by farther addition to x it decreaſes, till the product becomes nothing, and *vice verſa*.

To prove this, let $\dot{x}$ be a very ſmall number, ſo ſmall that its powers multiplied into any given number ſhall be leſs than any aſſignable number, and let $\dot{x}$ added to or taken from $x = y$.

To take the ſimpleſt caſe, let $a x - x^2 = k$ and $y = x + \dot{x}$.

$$\therefore a y = a x + a \dot{x}$$
$$y^2 = x^2 + 2 x \dot{x} + \dot{x}^2$$
$$\therefore a y - y^2 = a x - x^2 + a \dot{x} - 2 x \dot{x} \text{ nearly}$$
$$\therefore \overline{a y - y^2} \backsim \overline{a x - \dot{x}^2} = a \dot{x} \backsim 2 x \dot{x} \text{ nearly.}$$

If $y = x - \dot{x}$

$$a y = a x - a \dot{x}$$
$$y^2 = x^2 - 2 x \dot{x} + \dot{x}^2$$
$$\therefore a y - y^2 = a x - x^2 - a \dot{x} + 2 x \dot{x} \text{ nearly}$$
$$\therefore \overline{a y - y^2} \backsim \overline{a x - x^2} = 2 x \dot{x} \backsim a \dot{x} \text{ nearly.}$$

In the firſt caſe, by adding $\dot{x}$ the term $a y - y^2$ is greater than $a x - x^2$, if a is greater than $2 x$ or $\frac{a}{2}$ greater than x: but, if a is leſs than $2 x$, or $\frac{a}{2}$ leſs than x, then the term $a y - y^2$ is leſs than $a x - x^2$. Hence, the term

term $a\,x - x^2$ is increaſed by adding to x as long as x is leſs than $\frac{a}{2}$; but it is decreaſed by adding to x as long as x is greater than $\frac{a}{2}$.

In the ſecond caſe, by taking away x the term $a\,y - y^2$ is greater than $a\,x - x^2$, when $2\,x$ is greater than a, or x is greater than $\frac{a}{2}$: but, if $2\,x$ is leſs than a, or x leſs than $\frac{a}{2}$, then $a\,y - y^2$ is leſs than $a\,x - x^2$. Hence, the term $a\,x - x^2$ is decreaſed by taking from x as long as x is leſs than $\frac{a}{2}$; but it is increaſed by taking from x as long as x is greater than $\frac{a}{2}$.

If therefore in the equation $a\,x - x^2 = k$, d leſs than $\frac{a}{2}$ is one root, then no other number leſs than $\frac{a}{2}$ can be a root; for, by adding to or taking away from d numbers till $d \pm z$ is equal to $\frac{a}{2}$, the term $a\,x - x^2$ muſt equal a number greater or leſs than k. But ſince, by adding to d till x becomes equal to $\frac{a}{2}$, the term $a\,x - x^2$ is increaſed, and then by further addition is decreaſed till it becomes nothing, there muſt be another number which will make $a\,x - x^2$ equal to k: and there can be only one

one number greater than $\frac{a}{2}$; for by adding now to x, the term $a\,x - x^2$ is conſtantly diminiſhed.

The ſame may be proved generally.

Thus let $a\,x^n - x^{m+n} = k$.

$\therefore x^n \times \overline{a - x^m} = k$.

Let $y = x \pm \dot{x}$.

$\therefore y^n \times \overline{a - y^m} \mathrel{\text{ᔓ}} x^n \times \overline{a - x^m}$ is equal to the increaſe or decreaſe of $x^n \times \overline{a - x^m}$ by ſubſtituting a number for x but nearly equal to it in the equation $a\,x^n - x^{m+n} = k$.

$$y^n = x^n \pm n\,x^{n-1}\,\dot{x} \text{ nearly;}$$
$$a - y^m = a - x^m \mp m\,x^{m-1}\,\dot{x} \text{ nearly;}$$
$$\therefore y^n \times \overline{a - y^m} = a\,x^n \pm n\,a\,x^{n-1}\,\dot{x}$$
$$- x^{m+n} \mp n\,x^{m+n-1}\,\dot{x}$$
$$\mp m\,x^{m+n-1}\,\dot{x}$$

$\therefore y^n \times \overline{a - y^m} \mathrel{\text{ᔓ}} x^n \times \overline{a - x^m} = n\,a\,x^{n-1}\,\dot{x} \mathrel{\text{ᔓ}} n\,x^{m+n-1}\,\dot{x} + m\,x^{m+n-1}\,\dot{x} = x^{n-1}\,\dot{x} \times n\,a \mathrel{\text{ᔓ}} \overline{n+m} \cdot x^m$.

Hence, the term $x^n \times \overline{a - x^m}$ conſtantly increaſes or decreaſes, whilſt $n\,a$ is greater or leſs than $\overline{n+m} \cdot x^m$; but if it increaſes when $n\,a$ is greater than $\overline{n+m} \cdot x^m$, it decreaſes when $\overline{n+m} \cdot x^m$ is greater than $n\,a$: and, if it decreaſes when $n\,a$ is leſs than $\overline{n+m} \cdot x^m$, it increaſes when $\overline{n+m} \cdot x^m$ is leſs than $n\,a$.

Inſtances.

Inſtances.

$$8x - x^2 = 15.$$

The roots are 5 and 3.

Subſtitute 1, 2, 4, 6, 7, for x

$$\therefore 8x - x^2 = 2 \times \overline{8-2} = 12$$
$$8x - x^2 = 1 \times \overline{8-1} = 7$$
$$8x - x^2 = 4 \times \overline{8-4} = 16$$
$$8x - x^2 = 6 \times \overline{8-6} = 12$$
$$8x - x^2 = 7 \times \overline{8-7} = 7.$$

In this caſe $na = 8$ and $\overline{n+m} \cdot x^m = 2x$.

Therefore, when $n\ a$ is greater than $\overline{n+m} \cdot x^m$, 8 is greater than $2x$, or 4 is greater than x; and the numbers 1, 2, 3 ſubſtituted for x give reſults leſs than 4, but increaſing; 5, 6, 7 give alſo reſults leſs than 4, and decreaſing.

Let $\frac{61}{9} x^2 - x^3 = \frac{400}{9}$,

whoſe roots are 4 and 5;

let $x = 1$, then $\frac{61}{9} x^2 - x^3 = \frac{52}{9}$

$x = 2$ $\quad = \frac{172}{9}$

$x = 3$ $\quad = \frac{306}{9}$

$x = 6$ $\quad = \frac{252}{9}$

$x = 7$ Impoſſible.

The

The numbers 1, 2, 3, 4, give reſults increaſing; a number between 4 and 5 makes the reſult decreaſe: 5, 6 continue to make decreaſing reſults; if 7 is tried, there is no reſult.

Let $n = 1$ and $m = 2$.

$$\therefore a x^n - x^{m+n} = a x - x^2 = k.$$

Since this form admits of two roots, let them be c and d

$$\therefore a c - c^2 = k = a d - d^2$$
$$\therefore a c - a d = c^2 - d^2$$
$$\therefore a \times \overline{c - d} = c^2 - d^2$$
$$\therefore a = \frac{c^2 - d^2}{c - d} = c + d.$$

Hence, in an equation of this form a the copart of x is always equal to the ſum of the roots.

Since alſo $a = c + d$
$a c = c^2 + c d$
$\therefore a c - c^2 = c d$
but $a c - c^2 = k$
$\therefore k = c d$,

Hence k the known term is in this form always equal to the product of the roots. Thus let $12x - x^2 = 32$.

The roots are 8 and 4, and
$8 + 4 = 12 = a$, and $32 = 8 \times 4 = k$.

But in certain cafes equations in this form will only have one root; namely, when the root is one half of a. For let m be the number which makes $a\,x - x^2 = G$, or the greateft poffible number. Therefore, if x is made equal to $m + z$ or $m - y$, the product refulting will be lefs, and in both cafes may be equal to a number k.

$$\therefore a\,x - x^2 = \left\{\begin{matrix} a\,m + a\,z \\ -m^2 - 2\,m\,z - z^2 \end{matrix}\right\} = \left\{\begin{matrix} a\,m - a\,y \\ -m^2 + 2\,m\,y - y^2 \end{matrix}\right\} = k$$

$$\therefore a\,z + a\,y = 2\,m\,z + 2\,m\,y + z^2 - y^2$$

$$\therefore a \times \overline{z + y} = 2\,m \times \overline{z + y} + z^2 - y^2$$

$$\therefore a = 2\,m + z - y.$$

Now, fince m and a are conftant; but z or y may be taken at pleafure, $z - y$ muft be either an invariable number, or equal to nothing.

But $z - y$ cannot be an invariable number, for, by decreafing z and y, the number $z - y$ may be lefs than any affignable number. Therefore z muft be equal to y, and confequently $a = 2\,m$.

$$\therefore m = \frac{a}{2}.$$

Confequently, if the root of the equation $a\,x - x^2 = k$ is $\frac{a}{2}$ there can be only one root. In this cafe $a \times \frac{a}{2} - \frac{a^2}{4} = k = \frac{a^2}{4}$.

Therefore, if an equation of this form has only one root, that root is one half of the copart of x, and the known

known term k is the ſecond power of half the co-part a.

If $n = 1$ and $m = 3$; then
$a x^n - x^m = k = a x - x^3$.

Let m be the number which, placed for x, makes $a x - x^3 = G$ the greateſt poſſible. Therefore, as before, $m + z$ or $m - y$, ſubſtituted for x, will make $a x - x^3 = k$.

$$\therefore \left.\begin{array}{l} a x \left\{\begin{array}{l} = a m + a z \\ \\ = - m^3 - 3 m^2 z - 3 m z^2 - z^3 \end{array}\right\} = \\ - x^3 \end{array}\right.$$

$$\left.\begin{array}{l} a m - a y \\ - m^3 + 3 m^2 y - 3 m y^2 + y^3 \end{array}\right\} = k$$

$$\therefore a z + a y = 3 m^2 y + 3 m^2 z + 3 m z^2 - 3 m y^2 + z^3 + y^3$$

$$\therefore a \times \overline{z + y} = 3 m^2 \times \overline{y + z} + 3 m \times \overline{z^2 - y^2} + z^3 + y^3.$$

$$\therefore a = 3 m^2 + 3 m \times \overline{z - y} + \frac{z^3 + y^3}{z + y}.$$

But $3 m \times \overline{z - y} + \frac{z^3 + y^3}{z + y}$ muſt be equal to nothing.

$$\therefore a = 3 m^2$$

$$\therefore m = \frac{\sqrt{a}}{3}.$$

Example.

$$27 x - x^3 = 54.$$

In this caſe the equation can have only one root; for $x = 3 = \sqrt{\frac{27}{3}}$.

If x is made equal to one;

Then $27x - x^3 = 26$; and this equation will have another root $= 4{,}6235$ nearly.

$$\therefore y = 2 \text{ and } z = 1{,}6235$$
$$\text{but } y^3 + z^3 = 3m \times y^2 - z^2$$
$$\therefore 8 + \overline{1{,}6235}^3 = 9 \times \overline{4 - 1{,}6235}^2$$
$$\therefore \overline{1{,}6235}^2 \times \overline{9 + 1{,}6235} = 36 - 8 = 28$$
$$\therefore \overline{1{,}6235}^2 \times 10{,}6235 = 28.$$

Take the three firſt figures of each part of the compound term 1,62 and 1,06.

1,62	2,6244
1,62	1,06
324	157464
972	262440
162	2,781864
2,6244	

Had more figures been taken for z, the product would have approached nearer to 28.

In the ſame manner, if $n = 1$ and $m = 3$ that is if $ax - x^4 = k$, it may be diſcovered whether the equation has one or two roots. For if it has only one root, that root will be equal to $\sqrt[3]{\frac{a}{4}}$.

And in general it may be diſcovered whether an equation of the ſecond claſs has two roots or not. Thus

$$ax^n - x^{m+n} = k.$$

Let

Let M be the number which makes $a\,x_n - x^{m+n} = G$ or the greateſt poſſible number.

1ſt. Let $x = M + y$.

$$\therefore a\,x^n \left\{\begin{array}{l} a\,M^n + n\,a\,M^{n-1}\,y + n \cdot \frac{n-1}{2} a\,M^{n-2}\,y^2 \\ -M^{m+n} - \overline{m+n} . M^{m+n-1} y - \overline{m+n} . \frac{m+n-1}{2} \\ +\ldots\ldots\ldots \end{array}\right\} k$$
$$- x^{m+n} \left\{ M^{m+n-2}\,y^2 + . \right.$$

2d. Let $x = M - z$.

$$\therefore a\,x^n \left\{\begin{array}{l} a\,M^n + n\,a\,M^{n-1} z + n \cdot \frac{n-1}{2} a\,M^{n-2} \\ -M^{m+n} + \overline{m+n} . M^{m+n-1} z - \overline{m+n} . \\ z^2 + \ldots\ldots \\ \frac{m+n-1}{2} M^{m+n-2} z^2 + \ldots \end{array}\right\} k$$
$$- x^{m+n} \left\{ \right.$$

$\therefore n\,a\,M^{n-1} \times \overline{z+y} = \overline{m+n} . M^{m+n-1} \times \overline{z+y} + A\,z^2 - B\,z^3 + C . z^4$.

But $A\,z^2 - B z^3 + C\,z^4 \ldots\ldots = 0$

$\therefore n\,a\,M^{n-1} \times \overline{z+y} = \overline{m+n} . M^{m+n-1} \times \overline{z+y}$

$\therefore n\,a\,M^{n-1} = \overline{m+n} . M^{m+n-1}$

$$\therefore \frac{n\,a}{m+n} = M^m$$

$$\therefore M = \sqrt[m]{\frac{n\,a}{m+n}}.$$

Since

$$\text{ince } M = \sqrt[m]{\frac{n\,a}{m+n}}.$$

$$M^m = \frac{n\,a}{m+n}$$

$$\therefore a = M^m \times \frac{m+n}{n}.$$

But, since $a\,x^n - x^{m+n} = k$

$$a = \frac{k + x^{m+n}}{x^n}$$

That is, when $x = M$

$$a = \frac{k + M^{m+n}}{M^n} = M^m \times \frac{m+n}{n}$$

$$\therefore k + M^{m+n} = M^{m+n} \times \frac{m+n}{n}$$

$$\therefore k = M^{m+n} \times \frac{m+n}{n} - M^{m+n} = M^{m+n} \times \frac{m}{n}$$

To find the relation between a, k, and the roots of the equation $a\,x^n - x^{m+n} = k$.

Let the roots be c and d.

$$\therefore a\,c^n - c^{m+n} = k = a\,d^n - d^{m+n}$$

$$\therefore a\,c^n - a\,d^n = c^{m+n} - d^{m+n}$$

$$\therefore a \times \overline{c^n - d^n} = c^{m+n} - d^{m+n}$$

$$\therefore a = \frac{c^{m+n} - d^{m+n}}{c^n - d^n}$$

$$\therefore a = c^m + d^m + \frac{c^m\,d^n - d^m\,c^n}{c^n - d^n}.$$

Hence,

Hence, if m is equal to n, a is equal to the ſum of the m^{th} powers of the roots of the equation; for in this caſe $c^m d^n - d^m c^n = 0$.

If m is greater than n, a is greater than the ſum of the m^{th} powers of the roots. For in this caſe $\frac{c^m d^n - d^m c^n}{c^n - d^n}$ is a number to be added to $c^m + d^m$,

If m is leſs than n, then $d^m c^n$ is greater than $c^m d^n$, and a number equal to $\frac{d^m c^n - c^m d^n}{c^n - d^n}$ is to be taken away from $c^m + d^m$.

Inſtances.

1ſt. $m = n$.

$$12x - x^2 = 32.$$

The roots 8 and 4

$$a = c + d = 8 + 4 = 12.$$

2d. $m < n$

$$27x - x^3 = 26.$$

The roots are 1 and 4,6235 . . . .

$$a < 1 + 4{,}6235 \text{ or } 5{,}6235$$

3d. $m > n$

$$\frac{61}{9} x^2 - x^3 = \frac{400}{9}.$$

The roots are 4 and 5.

$$4 + 5 = 9 = \frac{81}{9} < \frac{61}{9} \text{ or } a.$$

Since

Since $a\,x^n - x^{m+n} = k$

$$a = \frac{k + x^{m+n}}{x^n} = \frac{k + c^{m+n}}{c^n} = \frac{k + d^{m+n}}{d^n}.$$

$$\therefore k\,d^n + c^{m+n}\,d^n = k\,c^n + d^{m+n}\,c^n$$

$$\therefore c^{m+n}\,d^n - d^{m+n}\,c^n = k\,c^n - k\,d^n$$

$$\therefore c^n\,d^n \times \overline{c^m - d^m} = k \times c^n - d^n$$

$$\therefore k = c^n\,d^n \times \frac{c^m - d^m}{c^n - d^n}.$$

Hence, if $m = n$, k is equal to the product of the n^{th} powers of the roots.

If m is greater than n, k is greater than the product of the n^{th} powers of the roots.

If m is leſs than n, k is leſs than the product of the n^{th} powers of the roots.

Inſtances.

1ſt, $12\,x - x^2 = 32$

The roots are 8 and 4.

$k = 32 = c\,d = 8 \times 4$.

Let $27\,x - x^3 = 26$.

The roots are 1 and 4,6235 and m is greater than n.

$k = 26 < 1 \times 4{,}6235$.

Let $\frac{61}{9}\,x^2 - x^3 = \frac{400}{9}$.

The

The roots are 4 and 5, and m is leſs than n; $k = \frac{400}{9} > 4^2 \times 5^2$ or 16×25.

To make equations of the form $a\,x^n - a^{m+n} = b$ with either one or two roots.

Make $a = c^m + d^m \pm c^n\,d^n \times \frac{c^{m-n} - d^{m-n}}{c^n - d^n}$.

and make $k = c^n\,d^n \times \frac{c^m - d^m}{c^n - d^n}$.

Inſtances.

Make an equation of the form $a\,x - x^2 = k$ with the two roots 7 and 9.

$$a = 7 + 9 = 16$$
$$k = 7 \times 9 = 63$$
$$\therefore 16\,x - x^2 = 63.$$

Make an equation of the form $a\,x - x^3 = k$ with the roots 7 and 2.

$$n = 1;\ m = 2.$$

$$a = c^m + d^m \frac{+ c^m\,d^n - d^m\,c^n}{c^n - d^n} = c^2 + d^2 + \frac{c^2\,d - d^2\,c}{c - d} =$$

$$c^2 + d^2 + c\,d \times \frac{c - d}{c - d} = c^2 + c\,d + d^2 = 49 + 14 + 4 = 67$$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c\,d \times \frac{c^2 - d^2}{c - d} = c\,d \times \overline{c + d}$$

$$= 14 \times 9 = 126$$

$$\therefore 67\,x - x^3 = 126.$$

Make an equation of the form $a\,x^2 - x^3 = k$ with the roots 5 and 2.

$$n = 2\,;\ m = 1.$$

$$a = c^m + d^m + \frac{c^m\,d^n - d^m\,c^n}{c^n - d^n} = c + d + \frac{c\,d^2 - d\,c^2}{c^2 - d^2}$$

$$= c + d - c\,d \times \frac{c - d}{c^2 - d^2} = c + d - \frac{c\,d}{c + d} =$$

$$5 + 2 - \frac{10}{7} = 7 - 1\tfrac{3}{7} = 5\tfrac{4}{7}$$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c^2\,d^2 \times \frac{c - d}{c^2 - d^2} = \frac{c^2\,d^2}{c + d} =$$

$$\frac{25 \times 4}{7} = \frac{100}{7} = 14\tfrac{2}{7}$$

$$\therefore 5\tfrac{4}{7}\,x^2 - x^3 = 14\tfrac{2}{7}.$$

Make an equation of the form $a\,x^3 - x^5 = k$ with the roots 7 and 6.

$$n = 3\,;\ m = 2.$$

$$a = c^m + d^m + \frac{c^m\,d^n - c^n\,d^m}{c^n - d^n} = c^2 + d^2 + \frac{c^2\,d^3 - c^3\,d^2}{c^3 - d^3}$$

$$= c^2 + d^2 - c^2\,d^2 \times \frac{c - d}{c^3 - d^3} = c^2 + d^2 - \frac{c^2\,d^2}{c^2 + c\,d + d^2}$$

$$= 49 + 36 - \frac{49 \times 36}{49 + 42 + 36} = 85 - \frac{1764}{127} =$$

$$85 - 13\tfrac{113}{127} = 71\tfrac{14}{127}$$

$$k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c^3 d^3 \times \frac{c^2 - d^2}{c^3 - d^3} = c^3 d^3 \times \frac{c + d}{c^2 + cd + d^2} =$$

$$7^3 \times 6^3 \times \frac{13}{49 + 42 + 36} = \frac{7^3 \times 6^3 \times 13}{127} = 7583\tfrac{103}{127}.$$

$$\therefore 71\tfrac{14}{127} x^3 - x^5 = 7583\tfrac{103}{127}.$$

Make an equation of the form $a x^2 - x^5 = k$, having only one root 3.

$$n = 2\,;\ m = 3.$$

$$M = \sqrt[m]{\frac{n a}{m + n}} = \sqrt[3]{\frac{2 a}{5}} = 3.$$

$$\therefore \frac{2 a}{5} = 3^3 = 27$$

$$\therefore a = \frac{135}{2} = 67{,}5$$

$$k = M^{m+n} \times \frac{m}{n}$$

$$\therefore k = 3^5 \times \frac{3}{2} = \frac{729}{2} = 364{,}5$$

$$\therefore 67{,}5\, x^2 - x^5 = 364{,}5.$$

The same might have been done from the general expreſſions for a and k in terms of c and d, by ſuppoſing that, when there is only one root, the two terms c and d approach nearer to equality than by any given difference. Thus

$$a = c^m + d^m + \frac{c^m d^n - c^n d^m}{c^n - d^n} = c^3 + d^3 + \frac{c^3 d^2 - c^2 d^3}{c^2 - d^2}$$

$$= c^3 + d^3 + c^2 d^2 \times \frac{c - d}{c^2 - d^2} = c^3 + d^3 + \frac{c^2 d^2}{c + d}$$

If d is fuppoſed to be nearly equal to c;

$$a = 2c^3 + \frac{c^4}{2c} = 2c^3 + \frac{c^3}{2} = \frac{5c^3}{2} = \frac{5 \cdot 27}{2} = 67{,}5.$$

Alſo $k = c^n d^n \times \frac{c^m - d^m}{c^n - d^n} = c^2 d^2 \times \frac{c^3 - d^3}{c^2 - d^2} =$

$c^2 d^2 \times \frac{c^2 + cd + d^2}{c + d}$ and d being fuppoſed nearly equal to c

$$k = c^4 \times \frac{3c^2}{2c} = c^4 \times \frac{3c}{2} = \frac{3 \cdot c^5}{2} = \frac{3 \cdot 3^5}{2} = 364{,}5.$$

In the ſame manner that the relation between a, k, and the roots of the equation $a x^n - x^{m+n} = k$ has been diſcovered; the relation between a, k, and the roots of the equations $x^{m+n} + a x^n = k$, and $x^{m+n} - a x^n = k$ may be inveſtigated. Thus let d be the root of the equation $x^{m+n} + a x^n = k$, and c the root of the equation $x^{m+n} - a x^n = k$; then

$$c^{m+n} - a c^n = k = d^{m+n} + a d^n.$$

$$\therefore c^{m+n} - d^{m+n} = a c^n + a d^n = a \times \overline{c^n + d^n}$$

$$\therefore a = \frac{c^{m+n} - d^{m+n}}{c^n + d^n} = c^m - \frac{c^m d^n}{c^n + d^n} - d^m + \frac{c^n d^m}{c^n + d^n}$$

$$= c^m - d^m - c^n d^n \times \frac{c^{m-n} - d^{m-n}}{c^n + d^n}.$$

Alſo, ſince $d^{m+n} + a\,d^n = k$, $a = \frac{k - d^{m+n}}{d^n}$; and ſince $c^{m+n} - a\,c^n = k$, $a = \frac{c^{m+n} - k}{c^n}$.

$$\therefore \frac{k - d^{m+n}}{d^n} = \frac{c^{m+n} - k}{c^n}$$

$$\therefore k\,c^n - d^{m+n}\,c^n = c^{m+n}\,d^n - k\,d^n$$

$$\therefore k \times \overline{c^n + d^n} = c^{m+n}\,d^n + d^{m+n}\,c^n = c^n\,d^n \times \overline{c^m + d^m}$$

$$\therefore k = c^n\,d^n \times \frac{c^m + d^m}{c^n + d^n}.$$

Hence, if $m = n = 1$, $a = c - d$ and $k = c\,d$.

Let $x^2 + 3x = 40 = y^2 - 3y$.

Here $c = 8$ and $d = 5$
$a = 3 \quad = 8 - 5$
$k = 40 = 8 \times 5$.

A relation alſo is to be found between the roots of the equation $x^{m+n} - a\,x^n = k$ and $a\,y^n - y^{m+n} = k$, which may ſometimes be of uſe.

$$x^{m+n} - a\,x^n = a\,y^n - y^{m+n}$$

$$\therefore x^{m+n} + y^{m+n} = a\,y^n + a\,x^n = a \times \overline{y^n + x^n}$$

$$\therefore a = \frac{x^{m+n} + y^{m+n}}{x^n + y^n}.$$

In certain caſes $x^{m+n} + y^{m+n}$ is diviſible by $x^n + y^n$; in which caſes the relation between x and the roots c and d of the equation $a\,y^n - y^{m+n}$ is eaſily diſcovered. Thus let

let $m = 2$, and $n = 1$.

$$\therefore a = \frac{x^3 + y^3}{x + y} = x^2 - yx + y^2 = x^2 - cx + c^2 =$$

$$x^2 - dx + d^2$$

$$\therefore c^2 - d^2 = cx - dx = x \times \overline{c - d}$$

$$\therefore x = c + d.$$

Hence, the root of the equation $x^3 - ax = k$ is equal to the fum of the roots of the equation $ax - x^3 = k$.

Alfo a relation may be found between k and the roots of thefe two equations. For

$$a x^n = x^{m+n} - k \quad \therefore a = \frac{x^{m+n} - k}{x^n}$$

$$\text{and } a y^n = y^{m+n} - k \quad \therefore a = \frac{y^{m+n} - k}{y^n}$$

$$\therefore \frac{x^{m+n} - k}{x^n} = \frac{c^{m+n} - k}{c^n} = \frac{d^{m+n} - k}{d^n}$$

$$\therefore x^{m+n} c^n - k c^n = x^n c^{m+n} - k x^n$$

$$\text{and } x^{m+n} d^n - k d^n = x^n d^{m+n} - k x^n$$

$$\therefore x^{m+n} \times \overline{c^n - d^n} - k \times \overline{c^n - d^n} = x^n \times \overline{c^{m+n} - d^{m+n}}$$

$$\therefore x^{m+n} - x^n \times \frac{c^{m+n} - d^{m+n}}{c^n - d^n} = k.$$

Let $n = 1$ and $m = 2$.

$$\therefore k = x^3 - x \times \frac{c^3 - d^3}{c - d} = x^3 - x \times \overline{c^2 + cd + d^2}$$

$$= x^3 - x \times \overline{c \times c + d + d^2} = x^3 - cx^2 - d^2 x = x^2$$

×

$\times \overline{x - c} - d^2 x = x^2 d - d^2 x = d x \times \overline{x - d} = c d x.$

Hence, the root of an equation $x^3 - a x = k$ is equal to the ſum of the roots of the equation $a x - x^3 = k$; and k is equal to the product of the roots of the two equations.

Inſtance.

$$x^3 - 76 x = 240.$$

The root of this equation is 10; and the roots of the equation $76 x - x^3 = 240$ are ſix and four, and $240 = 6 \times 4 \times \overline{6 + 4}$.

A ſimilar relation exiſts between the roots of the equations $x^{m+n} + a x^n = k$, $a y^n - y^{m+n} = k$ and a and k.

$$\text{For } x^{m+n} + y^{m+n} = a y^n - a x^n = a \times \overline{y^n - x^n}$$

$$\therefore a = \frac{x^{m+n} + y^{m+n}}{y^n - x^n}.$$

$$\text{Alſo } a = \frac{k - x^{m+n}}{x^n} = \frac{k + y^{m+n}}{y^n}$$

$$\therefore k y^n - x^{m+n} y^n = k x^n + y^{m+n} x^n$$

$$\therefore k \times \overline{y^n - x^n} = y^{m+n} x^n + x^{m+n} y^n$$

$$\therefore k = \frac{y^{m+n} x^n + x^{m+n} y^n}{y^n - x^n}.$$

To

To find the root or roots of the equation $a x^n - x^{n+s} = k$.

Firſt try, whether the equation has one or two roots by the expreſſions given for a and k. If it has only one root, that root is found from either of thoſe expreſſions. If it has two roots; try firſt, whether the leaſt root is a whole number, by applying numbers leſs than M already found: and having thus found the leaſt root, the other root may in ſome caſes be eaſily found by the expreſſions for a and k. If the leaſt root is not a whole number, take the difference between M and the neareſt whole number to it, which difference being added to M gives a number very near the greateſt root. By trial it will then be found, whether the greateſt root is a whole number; and, if it is, the leaſt root may alſo, in ſome caſes, be eaſily found from the expreſſions for a and k. If neither of the roots are whole numbers, they will be found beſt by the method of approach.

Let $39y - y^3 = 70$.

$$\therefore M = \sqrt{\frac{39}{3}} = \sqrt{13} = 3{,}6 \ldots .$$

$$M = \sqrt[3]{\frac{k}{2}}$$

$$\text{but } \sqrt[3]{\frac{70}{2}} = \sqrt[3]{35} = 3{,}2.$$

Hence, the equation has two roots.

$$y \doteq 3\ \ 2\ \ 5$$

$$\frac{70}{3} = 23\tfrac{1}{3},\ 35\ 14$$

$$39 - y^2 = 30\ \ 35\ 14.$$

By trying 3 according to the mode of dividers, it appears, that 3 is too great for the leaſt root: and on trying 2, it ſucceeds; conſequently 2 is one root, and as the greateſt root is not ſo far from M as the leaſt is, the greateſt root muſt be very near to if not five. Conſequently 5 was tried for the next ſtep, and the two roots are found, namely 2 and 5.

$$\text{Let } 25y - y^3 = 36.$$

$$M = \sqrt[3]{\frac{25}{3}} = \sqrt{8,3}$$

$$M = \sqrt[3]{\frac{k}{2}} = \sqrt[3]{\frac{36}{2}} = \sqrt[3]{18}.$$

But $\sqrt{8,3}$ is not equal to $\sqrt[3]{18}$. Therefore the equation has two roots

y	$= 2$	1	3	4
$\frac{36}{y}$	$= 18$	36	12	9
$25 - y^2$	$= 21$	24	16	9.

Hence, the leaſt root lies between one and two, and the greateſt root is 4. Therefore, ſince

$$a = c^m + d^m + \frac{c^m d^n - d^m c^n}{c^n - d^n}$$

$$25 = 4^2 + d^2 + \frac{4^2 d - 4 d^2}{4 - d} = 4^2 + d^2 + 4d$$

$$\therefore d^2 + 4d = 9$$

$$\therefore d = \sqrt{9+4} - 2 = \sqrt{13} - 2 = 1,60551\ldots$$

Let $9x^2 - x^3 = 100$ *.

$$M = \sqrt[m-n]{\frac{na}{m+n}} = \frac{2a}{3} = \frac{18}{3} = 6$$

$$M = \sqrt[m+n]{k \times \frac{n}{m}} = \sqrt[3]{100 \times \frac{2}{1}} = \sqrt[3]{200} = 5, \ldots$$

Hence, the equation has two roots

$$\text{Let } 9 - x = 4.$$
$$\frac{100}{9-x} = 25$$
$$x^2 = 25.$$

Hence, $x = 5$, and the greateſt root is not a whole number.

$$a = c^m + d^m + \frac{c^m d^n - d^m c^n}{c^n - d^n} = c + d + \frac{c d^2 - d c^2}{c^2 - d^2}$$

$$= c + d - cd \times \frac{c - d}{c^2 - d^2} = c + d - \frac{cd}{c+d} = \frac{c^2 + cd + d^2}{c + d}$$

$$= \frac{c^2 + 5c + 25}{c + 5} = 9$$

$$\therefore c^2 + 5c + 25 = 9c + 45$$
$$\therefore c^2 - 4c = 20$$
$$\therefore c = \sqrt{20 + 4} + 2 = 2 + \sqrt{24}.$$

Let $1000x - x^3 = 174$ †.

The leaſt root is evidently leſs than one; therefore x^3 bears a very ſmall proportion to $1000x$, and

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* Prob. XIII. XIV. Raphſon. Analyſis.

† Raphſon. XXXI.

1000

$$1000\,d = 174 \text{ nearly}$$

$$\therefore d = ,174;$$

$$\text{but } d = ,174 + \frac{x^3}{1000}$$

$$\therefore d = ,174 + \left|\frac{\overline{174}}{1000}\right|^3 \times \frac{1}{1000} \text{ nearly}$$

$$\text{Log. } \overline{174}^3 = 3.\ \text{Log. } 174 = 3 \cdot 2,2405492 = 6,7216476$$

$$\therefore \overline{174}^3 = 5,268022 \times 10^6 \text{ nearly}$$

$$\therefore \frac{\overline{174}^3}{\left|\overline{1000}\right|^3} \times \frac{1}{1000} = 5,268022 \times \frac{10^6}{10^9 \times 10^3} = \frac{5,268022}{10^6}$$

$$= ,000005268022$$

$$\therefore d = ,174005268022 + z.$$

For the greateſt root a very near approach may be made in a ſimilar manner.

$$1000\,c - c^3 = 174$$

$$\therefore 1000\,c - 174 = c^3$$

$$\therefore 1000 - \frac{174}{c} = c^2$$

$$\therefore c = \sqrt{1000} \text{ nearly}$$

$$\therefore c^2 = 1000 - \frac{174}{\sqrt{1000}} \text{ nearly}$$

$$\text{Log. } \frac{174}{\sqrt{1000}} = \left.\begin{array}{l}\text{Log. } 174 - \frac{1}{2} \cdot 3 = 2,2405492 \\ \qquad\qquad\qquad\quad - 1,5\end{array}\right\} =$$

$$= ,7405492$$

$$\therefore \frac{174}{\sqrt{1000}} = 5,502362$$

$\therefore$

$\therefore c^2 = 1000 - 5{,}502362 = 994{,}497637 \ldots$

$\therefore c = 31{,}5363$ nearly.

Let $77284\,x - x^3 = 8013128$. (RAPHSON, XI.)

Make $x = 100\,y$.

$\therefore 7{,}7284\,y - y^3 = 8{,}013128$

$$M = \sqrt{\frac{7{,}7284}{3}} = \sqrt{2{,}57613} = 1{,}605$$

$y =$ 1 1,2 1,4 18,3

$$\frac{8{,}013128}{y} = 8{,}013128\ \ 6{,}677\ \ 5{,}7236\ \ 4{,}378.$$

$$7{,}7284 - y^2 = 6{,}7284\ \ 6{,}2884\ \ 5{,}7684\ \ 4{,}3795.$$

The leaſt root appears to be between 1,2 and 1,4, but neareſt to 1,4 and nearly equal to 1,37; for 6,677 — 6,2884 = ,389 and 5,7236 — 5,7684 = ,044.

$\therefore \frac{,044 + ,389}{20} = \frac{,433}{20} = ,0216$ and $\frac{,044}{,0216} = 2, \ldots$

Therefore ſomewhat more than two hundred parts are to be taken from 1,4, to give the true value of d. Since $d = 1{,}37$ nearly, $M - d = ,23 \ldots \therefore M + ,23 = 1{,}835 = c$ nearly. Hence, 1,83 was made a divider, and it appears that c is ſomewhat greater than 1,83. Therefore for the firſt approach make $c = 1{,}83 + z$. Raphſon after a very long calculation makes x to be 179,79652, an error which aroſe from inattention to the marks + and —, an inattention, which is very frequently fatal in approaches to the roots of an equation in this form. This equa-

equation alſo Baron Maſeres ſolved by tabular ſines, which give for the root c. 179,7368 *. But on the error having been pointed out to him, he ſolved it immediately by the method of approach, and confirmed the truth of this ſtatement.

Let $300x - x^3 = 1000$. (RAPHSON, XII.)

$$300d = 1000 + d^3 = 1000 \text{ nearly}$$

$$\therefore d = \frac{1000}{300} = \frac{10}{3} \text{ nearly}$$

$$\therefore d = \frac{10}{3} + \frac{10^3}{3^3 \cdot 300}$$

$$\therefore d = \frac{10}{3} + \frac{1000}{27 \cdot 300} = \frac{10}{3} + \frac{10}{3} \times \frac{1}{27} = \frac{10}{3} \times 1 + \frac{1}{27} = \frac{10}{3} \times \frac{28}{27} = \frac{280}{9 \cdot 9} = \frac{31,111}{9} = 3,45679012 \ldots .$$

$$\therefore d = \frac{10}{3} + \frac{\overline{3,45679.}^3}{300} = \frac{10}{3} + \frac{41,30665}{300} = 3,333 + ,13768883 = 3,47102216$$

$$\therefore d = \frac{10}{3} + \frac{\overline{3,47102216}^3}{300} = \frac{10}{3} + \frac{41,81884}{300} = 3,333 + ,3939613 = 3,47272946.$$

In the ſame manner approaches might be continually made, but they are very ſlow, ſince x^3 does not bear a

* Maſere's Diſſertation on the Uſe of the Negative Sign, page 255.

ſmall

ſmall proportion to 1000. Let this equation be tried by dividers.

$$M = \sqrt{\frac{300}{3}} = \sqrt{100} = 10$$

x	$=5$	3	4	3,5	15	16	
$\frac{1000}{x}$	$=200$	333	,250	285,5	66,β	62,5	
$300-x^2$	$=275$	291	284	287,75	75	44.	

Hence, d is between 3,5 and 4, and is equal to $3{,}47 + z$ and c is between 15 and 16, and is equal to $15{,}3 + z$. In the former caſe, $d = 3{,}472963553338$, in the latter, $c = 15{,}3208887 \ldots$ nearly

Let $5x - x^3 = 4$.

Here d is evidently equal to one. Therefore a which is equal to $c^2 + cd + d^2$ is equal to $c^2 + c + 1$.

$$\therefore 5 = c^2 + c + 1$$
$$\therefore c^2 + c = 4$$
$$\therefore c = \sqrt{4+\tfrac{1}{4}} - \tfrac{1}{2} = \frac{\sqrt{17}-1}{2}$$ *

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* App. page 327.

CHAP.

CHAP. IV.

EQUATIONS OF THE THIRD CLASS.

EQUATIONS of the third claſs have on the unknown ſide three terms; and in one of their forms may have three roots, in others two roots, and in others can have only one root.

Forms of this Claſs.

$$\text{I.} \begin{cases} 1 \cdot x^m + a\,x^n + b x^o = c \\ 2 \cdot x^m + a\,x^n - b\,x^o = c \\ 3 \cdot x^m - a\,x^n - b\,x^o = c \end{cases}$$

$$\text{II.} \begin{cases} 1 \cdot a\,x^n + b\,x^o - x^m = c \\ 2 \cdot a\,x^n - b\,x^o - x^m = c \\ 3 \cdot b\,x^o - a\,x^n - x^m = c \end{cases}$$

$$\text{III.} \quad x^m + b\,x^o - a\,x^n = c$$

The equations in the firſt order of forms of this claſs correſpond with thoſe of either the firſt or ſecond forms of equations of the ſecond claſs, and conſequently can have but one root. The ſecond order of equations correſponds with the third form of equations of the ſecond claſs, and conſequently equations of either of the forms in this order may have two roots. The third order has only one form, and this form correſponds, in one caſe, with the ſecond form of the ſecond claſs, and conſequently

quently can have but one root; in the other caſe it correſponds with the third form of equations of the ſecond claſs, in which conſequently it may have two roots. And, as both caſes take place ſometimes in the ſame equation, this form admits of three roots.

To make theſe propoſitions eaſier to the learner, they ſhall be firſt proved in equations of the ſimpleſt kinds in this claſs; namely, when m is equal to 3, n equal to 2, and o equal to one.

The equations of the firſt order become in this caſe,

$$1 \cdot x^3 + a x^2 + b x = k$$
$$2 \cdot x^3 + a x^2 - b x = k$$
$$3 \cdot x^3 - a x^2 - b x = k.$$

1. $x^3 + a x^2 + b x = c$. This equation can evidently have only one root: for, if any number is added to or taken away from x, the ſum on the unknown ſide is made greater or leſs than it was before, and conſequently muſt be greater or leſs than k.

2. $x^3 + a x^2 - b x = k$. Since $x^3 + a x^2$ is greater than $b x$, and by adding to x, the increaſe of $x^3 + a x^2$ is greater than that of $b x$, the difference of the terms on the unknown ſide is increaſed by adding to x, and for the ſame reaſon is decreaſed by taking away from x. Conſequently by adding to x, the difference is made greater than k, by taking away from x the difference is made leſs than k.

3.

3. $x^3 - a x^2 - b x = k$. In this caſe the increaſe of x^3 is greater than that of $a x^2 + b x$; therefore by adding to x the difference of the terms is made greater than k, and by taking from x on the ſame principle the difference is made leſs than k.

The ſecond order of equations in this claſs becomes, when m, n and o are reſpectively equal to 3, 2 and 1,

1. $a x^2 + b x - x^3 = k$
2. $a x^2 - b x - x^3 = k$
3. $b x - a x^2 - x^3 = k$.

1. $a x^2 + b x - x^3 = k$. In an equation of this form the unknown ſide does not always increaſe with the increaſe of x, nor decreaſe with its decreaſe.

Let x become $x + \dot{x}$, $\dot{x}$ being a very ſmall number, then

$$\left.\begin{matrix} a x^2 \\ + b x \\ - x^3 \end{matrix}\right\} \begin{matrix} a x^2 + 2 a x \dot{x} + \dot{x}^2 \\ = + b x + b \dot{x} \\ - x^3 - 3 x^2 \dot{x} - 3 x \dot{x}^2 - \dot{x}^3 \end{matrix} = \left\{\begin{matrix} a x^2 + 2 a x \dot{x} \\ + b x + b \dot{x} \text{ nearly} \\ - x^3 - 3 x^2 \dot{x}. \end{matrix}\right.$$

$\therefore 2 a x \dot{x} + b \dot{x} \infty 3 x^2 \dot{x}$ is the increaſe or decreaſe nearly of the unknown ſide of the equation. If $2 a x + b$ is greater than $3 x^2$, the equation is increaſed; if $2 a x + b$ is leſs than $3 x^2$, the equation is decreaſed.

Now let $\dot{x}$ be taken from x, or x be made $x - \dot{x}$, then

$$\left.\begin{matrix} a x^2 \\ + b x \\ - x^3 \end{matrix}\right\} \begin{matrix} a x^2 - 2 a x \dot{x} + \dot{x}^2 \\ = + b x - b \dot{x} \\ - x^3 + 3 x^2 \dot{x} - 3 x \dot{x}^2 + \dot{x}^3 \end{matrix} = \left\{\begin{matrix} a x^2 - 2 a x \dot{x} \\ + b x - b \dot{x} \text{ nearly} \\ - x^3 + 3 x^2 \dot{x} \end{matrix}\right.$$

Hence, $3\,x^2\,\dot{x} \backsim \overline{2\,a\,x\,\dot{x} + b\,\dot{x}}$ is the increaſe or decreaſe nearly of the equation. If $3\,x^2$ is greater than $2\,a\,x + b$, the equation is increaſed; if $3\,x^2$ is leſs than $2\,a\,x + b$, the equation is decreaſed. As long therefore as $2\,a\,x + b$ is greater than $3\,x^2$, the unknown ſide of the equation is increaſed by adding to x and decreaſed by taking away from x; and if it is equal to k in this ſtate, it muſt be equal to k again; for as it increaſes till $2\,a\,x + b$ is equal to $3\,x^2$, from that time the unknown ſide decreaſes by adding to x. And it conſtantly decreaſes from a number greater than k to nothing.

2. $a\,x^2 - b\,x - x^3 = k$. In this the unknown ſide increaſes or decreaſes by the term $2\,a\,x\,\dot{x} \backsim \overline{b\,\dot{x} + 3\,x^2\,\dot{x}}$ when $\dot{x}$ is added to x, or by the term $\overline{b\,\dot{x} + 3\,x^2\,\dot{x}} \backsim 2\,a\,x\,\dot{x}$, if $\dot{x}$ is taken from x, and the reaſoning is as before.

3. $b\,x - a\,x^2 - x^3 = k$. In this form the unknown ſide increaſes or decreaſes by the term $\overline{b\,\dot{x} + 2\,a\,x\,\dot{x}} \backsim 3\,x^2\,\dot{x}$ by adding to x or by the term $2\,\overline{a\,x\,\dot{x} + 3\,x^2\,\dot{x}} \backsim b\,\dot{x}$ when $\dot{x}$ is taken from x, and the reaſoning is the ſame as above.

III. $x^3 + b\,x - a\,x^2 = k$.

In this form the unknown ſide may firſt increaſe by adding to x, then decreaſe, and laſtly be capable of increaſe without limit. As before, it increaſes if $3\,x^2\,\dot{x} + b\,\dot{x}$ is greater than $2\,a\,x\,\dot{x}$, it decreaſes when $3\,x^2\,\dot{x} + b\,\dot{x}$ is leſs than $2\,a\,x\,\dot{x}$, and as x may be increaſed without

without limit, $3 x^2 \dot{x} + b \dot{x}$ muſt, by continually adding to x, become again greater than $2 a x \dot{x}$ or equal to any aſſignable number. Hence, if the unknown ſide is equal to k when $3 x^2 \dot{x} + b x \dot{x}$ is greater than $2 a x \dot{x}$, it may again become equal to k when $2 a x \dot{x}$ is greater than $3 x^2 \dot{x} + b x \dot{x}$, and again equal to k when $3 x^2 \dot{x} + b x \dot{x}$ becomes again greater than $2 a x \dot{x}$.

But an equation in this form cannot have more than three roots, for in either ſtate whether $2 a x \dot{x}$ is greater or leſs than $3 x^2 \dot{x} + b x \dot{x}$, the unknown ſide conſtantly increaſes or decreaſes between the limits $x = o$ or x being ſuch that $3 x^2 \dot{x} + b x \dot{x} = 2 a x \dot{x}$.

Let d, e, f be the roots of an equation in this form, d being greater than e, and e greater than f.

$$\text{Since } x^3 + b x - a x^2 = k$$
$$d^3 + b d - a d^2 = k$$
$$\text{Alſo } e^3 + b e - a e^2 = k.$$
$$\text{Therefore } d^3 - e^3 + b \times \overline{d - e} = a \times \overline{d^2 - e^2}$$
$$\therefore d^2 + e d + e^2 + b = a \times \overline{d + e}$$
$$\text{Alſo } d^2 + d f + f^2 + b = a \times \overline{d + f}$$
$$\therefore d \times \overline{e - f} + e^2 - f^2 = a \times \overline{e - f}$$
$$\therefore d + e + f = a.$$

Hence, in equations of this form having three roots, a the copart of the ſecond power of x is equal to the ſum of the roots.

Since $d^2 + e\,d + e^2 + b = a \times \overline{d + e}$ by ſubſtituting for a its equal the ſum of the roots.

$$d^2 + e\,d + e^2 + b = \overline{d + e + f} \times \overline{d + e}$$
$$\therefore d^2 + e\,d + e^2 + b = d^2 + d\,e + d\,e + e^2 + d\,f + e\,f$$
$$\therefore b = d\,e + d\,f + e\,f.$$

Hence b, the copart of the ſimple unknown number, is always equal to the ſums of the products of each pair of the roots.

Now $d^3 + b\,d - a\,d^2 = k$. Therefore by ſubſtituting for b and a their equals in terms of the roots

$$d^3 + \overline{d\,e + d\,f + e\,f} \times d - d^2 \times \overline{d + e + f} = k$$
$$\therefore d^3 + d^2\,e + d^2 f + d\,e f - d^3 - d^2\,e - d^2 f = k$$
$$\therefore d\,e f = k.$$

Hence the known term is always the product of the roots multiplied together.

If one of the roots of an equation in this form having three roots is known, the other two may be found by means of an equation of the ſecond order. For

$$d^2 + e\,d + e^2 + b = a \times \overline{d + e}$$
Alſo $d^2 + f\,d + f^2 + b = a \times \overline{d + f}$
and $e^2 + e\,f + f^2 + b = a \times \overline{e + f}$.

Hence, if the leaſt root f is known, to find the next root e, the equation

e^2

$$e^2 + ef + f^2 + b = a \times e + f$$

would be employed.

And to find the greateſt root d, the equation

$$d^2 + df + f^2 + b = a \times \overline{d + f}$$

would be employed.

But they are both of the ſame form with the ſame coparts to the unknown number, and may be written thus:

$$x^2 + fx + f^2 + b = a \times \overline{f + x}$$

$$\therefore b + f^2 - af = ax - fx - x^2$$

$$\text{or } b + f^2 - af = x \times \overline{a - f} - x^2$$

Now this equation has two roots, whoſe ſum is equal to $a - f$ that is $d + e$ and their product is equal to $b + f^2 - af$ or to $b - f \times \overline{a - f}$ that is $b - f \times \overline{d + e}$ or $de + df + ef - df - ef$ that is to de.

Therefore d and e are the roots of the equation $x^2 + fx + f^2 + b = a \times \overline{f + x}$.

In the ſame manner if e had been known, the two roots d and f might be found from the equation

$$x^2 + ex + e^2 + b = a \times \overline{e + x}.$$

Or if d had been known, the other two roots might be found from the equation

$$x^2 + dx + d^2 + b = a \times \overline{d + x}.$$

Inſtance.

Inſtance.

Let $x^3 - 6x^2 + 11x = 6$ whoſe roots are 1, 2, 3. Suppoſe the leaſt root to have been the only one known. Then

$$x^2 + fx + f^2 + b = a \times \overline{f + x} \text{ becomes}$$

$$x^2 + x + 1 + 11 = 6 \times \overline{1 + x}$$

$$\therefore 12 - 6 = 6x - x - x^2$$

$$\therefore 6 = 5x - x^2$$

$$\therefore \sqrt{\frac{25}{4} - 6} = \begin{cases} \frac{5}{2} - x \\ \text{or } x - \frac{5}{2} \end{cases}$$

$$\text{but } \sqrt{\frac{25}{4} - 6} = \sqrt{\frac{25 - 24}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\therefore \frac{1}{2} = \frac{5}{2} - x$$

$$\therefore x = \frac{5}{2} - \frac{1}{2} = \frac{4}{2} = 2$$

$$\text{or } \frac{1}{2} = x - \frac{5}{2}$$

$$\therefore x = \frac{5}{2} + \frac{1}{2} = \frac{6}{2} = 3.$$

The reader may try the ſame proceſs from either of the other two roots being ſuppoſed the only one known.

If the equation has only two roots, then e and f or d and e are equal. In the firſt caſe if d is known, the other root is found by making $a - d = 2e$ $\therefore e = \frac{a - d}{2}$.

Or

Or if e in the ſecond caſe is known, $a - 2e = d$, which gives eaſily the number d. Thus, let

$$x^3 + 72x - 15x^2 = 112;$$

the roots are 7 and 4.

If 7 only had been known, then the root $e = \frac{a-d}{2} = \frac{15-7}{2} = 4$.

If 4 had been the root known,

$$d = a - 2e = 15 - 8 = 7.$$

In the ſame manner one root may be found from the other by means of the expreſſions b and k. For $k = de^2$, that is when $e = f$, $k = de^2$

$$\therefore d = \frac{k}{e^2}$$

$$\therefore e = \sqrt{\frac{k}{d}}$$

$$d = \frac{112}{16} = 7$$

$$e = \sqrt{\frac{112}{7}} = 4$$

Alſo $b = 2de + e^2$

$$\therefore d = \frac{b - e^2}{2e}$$

$$\text{and } e = \sqrt{b + d^2} - d$$

If the equation $x^3 + bx - ax^2 = c$ has only two roots de, the equation

d^2

$$d^2 + e\,d + e^2 + b = a \times \overline{d + e}$$

has only one root, namely $\frac{a - d}{2}$ equal to e. For

$$d^2 + b - a\,d = a\,e - e\,d - e^2 = e \times \overline{a - d} - e^2$$

and ſince the two roots of the above equation are equal,

$$e = \frac{a - d}{2}.$$

Subſtitute $\frac{a - d}{2}$ for e in the equation $d^2 + b - a\,d = e \times \overline{a - d} - e^2$.

$$\therefore d^2 + b - a\,d = \frac{a - d}{2} \times \overline{a - d} - \overline{\frac{a - d}{2}}^2 = \overline{a^2 - 2\,a\,d + d^2} \times \tfrac{1}{4}$$

$$\therefore 4\,d^2 + 4\,b - 4\,a\,d = a^2 - 2\,a\,d + d^2$$

$$\therefore 3\,d^2 - 2\,a\,d = a^2 - 4\,b$$

$$\therefore d^2 - \frac{2\,a\,d}{3} = \frac{a^2 - 4\,b}{3}$$

$$\therefore d - \frac{a}{3} = \sqrt{\frac{a^2}{9} + \frac{a^2 - 4\,b}{3}} = \sqrt{\frac{4a^2 - 4\,b \cdot 3}{9}}$$

$$\therefore d = \frac{a + 2\sqrt{a^2 - 3\,b}}{3}.$$

Alſo $e = \frac{a - d}{2}$ $\therefore e = \frac{1}{2} \cdot a - \frac{a + 2\sqrt{a^2 - 3\,b}}{3}$

$$\therefore e = \tfrac{1}{2} \times \frac{2\,a - 2\sqrt{a^2 - 3\,b}}{3} = \frac{a - \sqrt{a^2 - 3\,b}}{3}$$

Hence, if the equation $x^3 + bx - a\,x^2 = c$ has only two roots as above deſcribed, the ſecond power of a muſt be greater than three times b.

Inſtance.

Inftance.

Let the equation be $x^3 - 5x^2 + 7x = 3$ having two roots.

$$\therefore e = \frac{a - \sqrt{a^2 - 3b}}{3} = \frac{5 - \sqrt{25 - 21}}{3} = \frac{5 - 2}{3}$$

$= 1$ and $d = \frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{5 + 4}{3} = \frac{9}{3}$

$= 3$.

Since the equation $e^2 + b - ae = x \times \overline{a - e} - x^2$ gives two values of x, which are the roots d and f, by making either of theſe values equal to e, the roots of the equation $x^3 - ax^2 + bx = k$, when it has only two roots, will be found. Thus

$$x \times \overline{a - e} - x^2 = e^2 + b - ae$$

$$\therefore x = \frac{a - e}{2} = \sqrt{\left(\frac{a - e}{2}\right)^2 - e^2 - b + ae} =$$

$$\frac{\sqrt{a^2 - 2ae + e^2 - 4e^2 - 4b + 4ae}}{2} =$$

$$\frac{\sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

$$\therefore x = \frac{a - e \pm \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

That is, $d = \frac{a - e + \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$

and $f = \frac{a - e - \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$

Now let the equation $x^3 - a x^2 + b x = k$ have only two roots, then d or f may be made equal to e. Firſt let $d = e$.

$$\text{Then } e = \frac{a - e + \sqrt{a^2 + 2ae - 3e^2 - 4b}}{2}$$

$$\therefore 3e - a = \sqrt{a^2 + 2ae - 3e^2 - 4b}$$

$$\therefore 9e^2 - 6ae + a^2 = a^2 + 2ae - 3e^2 - 4b$$

$$\therefore 12e^2 = 8ae - 4b$$

$$\therefore 2ae - 3e^2 = b$$

$$\therefore \frac{2ae}{3} - e^2 = \frac{b}{3}$$

$$\therefore \frac{a}{3} \mp e = \sqrt{\frac{a^2}{9} - \frac{b}{3}}$$

$$\therefore e = \frac{a + \sqrt{a^2 - 3b}}{3}.$$

The other expreſſion $\frac{a - \sqrt{a^2 - 3b}}{3}$ cannot be applied in this inſtance, ſince e being equal to d muſt be greater than $\frac{a}{3}$.

$$f = a - 2e = a - \frac{2a + 2\sqrt{a^2 - 3b}}{3} = \frac{a - 2\sqrt{a^2 - 3b}}{3}.$$

Hence if the equation has two roots, and the greater is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$, the ſmaller will be equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$.

As

As f alſo may be equal to e, and f is equal to $\frac{a-e-\sqrt{a^2+2ae-3e^2-4b}}{2}$ then

$$e=\frac{a-e-\sqrt{a^2+2ae-3e^2-4b}}{2}.$$

$$\therefore a-3e=\sqrt{a^2+2ae-3e^2-4b}$$

$$\therefore a^2-6ae+9e^2=a^2+2ae-3e^2-4b$$

$$\therefore 8ae-12e^2=4b$$

$$\therefore \frac{2ae}{3}-e^2=\frac{b}{3}$$

$$\therefore e=\frac{a-\sqrt{a^2-3b}}{3}.$$

The other expreſſion $\frac{a+\sqrt{a^2-3b}}{3}$ cannot in this caſe be applied ſince a is greater than $3e$.

$d=a-2e=a-\frac{2a-2\sqrt{a^2-3b}}{3}=\frac{a+2\sqrt{a^2-3b}}{3}$. Hence, if the equation has two roots, and the ſmaller root is equal to $\frac{a-\sqrt{a^2-3b}}{3}$ the greater root muſt be equal to $\frac{a+2\sqrt{a^2-3b}}{3}$.

Thus, if the equation $x^3-ax^2+bx=k$ has only two roots, the one muſt be equal to $\frac{a+\sqrt{a^2-3b}}{3}$, and

the other to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$, or the one muſt be equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$, and the other $\frac{a + 2\sqrt{a^2 - 3b}}{3}$.

If the equation $x^3 + bx - ax^2 = c$ has three roots, e muſt be leſs than $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ but greater than $\frac{a - \sqrt{a^2 - 3b}}{3}$ and f leſs than $\frac{a - \sqrt{a^2 - 3b}}{3}$.

Inſtance.

$$x^3 - 12x^2 + 41x = 42$$

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{12 - \sqrt{144 - 123}}{3} = \frac{12 - \sqrt{21}}{3}$$

$$= \frac{12 - 4{,}5824\ldots}{3} = \frac{7{,}4175}{3} = 2{,}472.$$

The neareſt whole number 2 being tried ſucceeds, and conſequently there muſt be three roots.

$$\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{12 + 2.4{,}5824\ldots}{3} = \frac{12 + 9{,}1648}{3}$$

$$= \frac{12 + 9{,}1648}{3} = \frac{21{,}1648}{3} = 7{,}054\ldots$$

Hence e muſt lie between 2,472 and 7,054. Alſo it muſt be leſs than $\frac{a - f}{2}$ or $\frac{12 - 2}{2}$ or 5. Conſequently 3 being the next whole number to 2,472 is tried and ſucceeds.

ceeds. The third root therefore being equal to $a - e - f$, or $12 - 3 - 2$ muſt be ſeven.

If the equations are of the forms admitting only of two roots, by applying the ſame mode of reaſoning one root may be diſcovered from the other. Thus let the equation having two roots be of this form

$$a x^2 + b x - x^3 = k$$

then $a d^2 + b d - d^3 = k$

and $a e^2 + b e - e^3 = k$

$$\therefore a \times \overline{d^2 - e^2} + b \times \overline{d - e} = d^3 - e^3$$

$$\therefore a \times \overline{d + e} + b = d^2 + e d + e^2$$

$$\therefore a d^2 + a d e + b d = d^3 + e d^2 + e^2 d$$

$$\therefore a d^2 + b d - d^3 = e d^2 + e^2 d - a d e = d e \times \overline{d + e - a}$$

But $a d^2 + b d - d^3 = k$

$$\therefore k = d e \times \overline{d + e - a}.$$

Hence the ſum of the roots muſt be greater than a.

Since $a d^2 + b d - d^3 = k$

$$b = \frac{k - a d^2 + d^3}{d} = \frac{d e \times \overline{d + e - a} - a d^2 + d^3}{d}$$

$$= e \times \overline{d + e - a} - a d + d^2 = e d + e^2 - a e - a d + d^2$$

Let $7 x^2 + 14 x - x^3 = 120$.

One of the roots is 5.

$$k = d e \times \overline{d + e - a} = 5 d \times \overline{5 + d - 7} = 120$$

$$\therefore 25 d + 5 d^2 - 35 d = 120$$

$$\therefore d^2 - 2 d = 24$$

$$\therefore d = \sqrt{\overline{24 + 1}} + 1 = 5 + 1 = 6.$$

Alſo

$$\text{Alſo } b = d \times \overline{d + e - a} - e \times \overline{a - e}$$

$$\therefore 14 = 6 \times \overline{6 + 5 - 7} - 5 \times 2 = 6 \times 4 - 10.$$

In the ſame manner the expreſſion for b and k are diſcovered in the form

$$a x^2 - b x - x^3 = k$$

$$\text{Thus } a d^2 - b d - d^3 = k$$

$$a e^2 - b e - e^3 = k$$

$$\therefore a \times \overline{d^2 - e^2} - b \times \overline{d - e} = d^3 - e^3$$

$$\therefore a \times \overline{d + e} - b = d^2 + e d + e^2$$

$$\therefore a d^2 + a d e - b d = d^3 + e d^2 + e^2 d$$

$$\therefore a d^2 - b d - d^3 = e d \times \overline{d + e} - a d e$$

$$\text{But } a d^2 - b d - d^3 = k$$

$$\therefore k = e d \times \overline{d + e - a}.$$

$$\text{Alſo } b = \frac{a d^2 - d^3 - k}{d} = \frac{a d^2 - d^3 - e d \times \overline{d + e - a}}{d}$$

$$= a d - d^2 - e d - e^2 + a e = e \times \overline{a - e} - d \times \overline{d + e - a}.$$

Hence, in this form alſo the ſum of the roots is greater than a.

For the form $b x - a x^2 - x^3 = k$ the inveſtigation is of the ſame kind.

$$b d - a d^2 - d^3 = k$$

$$b e - a e^2 - e^3 = k$$

$$\therefore b \times \overline{d - e} - a \times \overline{d^2 - e^2} = d^3 - e^3$$

$$\therefore b - a \times \overline{d + e} = d^2 + e d + e^2$$

$$\therefore b d - a d^2 - a d e = d^3 + e d^2 + e^2 d$$

$$\therefore b\,d - a\,d^2 - d^3 = e\,d \times e + d + a$$

But $b\,d - a\,d^2 - d^3 = k$

$$\therefore k = e\,d \times \overline{e + d + a}$$

$$\text{Alſo } b = \frac{k + a\,d^2 + d^3}{d} = \frac{e\,d \times \overline{e + d + a} + a\,d^2 + d^3}{d}$$

$$= e^2 + e\,d + a\,e + a\,d + d^2 = d \times \overline{a + d + e} + e \times \overline{a + e}.$$

Hence in this form a does not depend at all on the ſum of the roots, and the equation may be made of any two numbers whatever for roots. In either of the forms alſo if one root is found the other may be found by means of the expreſſions for k and b in an equation of two terms.

If there is only one root to equations in theſe forms, it may be diſcovered by means of an equation of the ſecond claſs. For in this caſe d and e become equal, and for the equations $a\,x^2 + b\,x - x^3 = k$

$$b = d \times \overline{d + e - a} - e \times \overline{a - c}$$

That is, if $d = e$

$$b = d^2 + d^2 - a\,d - a\,d + d^2$$

$$\therefore b = 3\,d^2 - 2\,a\,d$$

$$\therefore \frac{b}{3} = d^2 - \frac{2\,a\,d}{3}$$

$$\therefore d = \sqrt{\frac{a^2}{9} + \frac{b}{3}} + \frac{a}{3} = \frac{a + \sqrt{a^2 + 3\,b}}{3}.$$

Hence, if an equation in this form has two roots, the greateſt muſt be greater than $\frac{a + \sqrt{a^2 + 3\,b}}{3}$ and the leaſt

leaſt muſt be leſs than $\frac{a + \sqrt{a^2 + 3b}}{3}$.

In the equation $a x^2 - b x - x^3 = k$

$$b = e \times \overline{a - e} - d \times \overline{d + e - a}$$

That is, if $d = e$

$$b = a d - d^2 - d^2 - d^2 + a d = 2 a d - 3 d^2$$

$$\therefore \frac{b}{3} = \frac{2 a d}{3} - d^2$$

$$\therefore \frac{a}{3} - d = \begin{cases} \sqrt{\frac{a^2}{9} - \frac{b}{3}} = \frac{\sqrt{a^2 - 3b}}{3} \\ \text{or } d - \frac{a}{3} = \end{cases}$$

$$\therefore d = \frac{a \pm \sqrt{a^2 - 3b}}{3}$$

Since $2d$ is greater than a, d is greater than $\frac{a}{2}$, much more therefore greater than $\frac{a}{3}$, or a number leſs than $\frac{a}{3}$. But $\frac{a - \sqrt{a^2 - 3b}}{3}$ is leſs than $\frac{a}{3}$; and conſequently this expreſſion cannot be uſed for d in the given equation: and if there is only one root, that root is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$.

Hence, if the equation has two roots, the greateſt root is greater than $\frac{a + \sqrt{a^2 - 3b}}{3}$ and the leaſt root is leſs than $\frac{a + \sqrt{a^2 - 3b}}{3}$.

In

In the equation $bx - x^2 - x^3 = k$

$$b = d \times \overline{a + d + e} + e \times \overline{a + e}$$

That is, if $d = e$

$$b = ad + d^2 + d^2 + ad + d^2 = 3d^2 + 2ad$$

$$\therefore \frac{b}{3} = d^2 + \frac{2ad}{3}$$

$$\therefore d = \sqrt{\frac{a^2}{9} + \frac{b}{3}} - \frac{a}{3} = \frac{\sqrt{a^2 + 3b} - a}{3}$$

Hence, if the equation has two roots the one is greater and the other leſs than $\frac{\sqrt{a^2 + 3b} - a}{3}$.

The enquiry into the roots of any equations in theſe forms is thus confined within certain limits, which, with the limits preſcribed by the equation itſelf, render the diſcovery of the roots oftentimes very eaſy.

Let $7x^2 + 14x - x^3 = 120$.

$$\therefore x \times \overline{7x + 14 - x^2} = 120$$

$\therefore 7x + 14$ is greater than x^2

Alſo $d + e$ is greater than 7

And d is greater than $\frac{a + \sqrt{a^2 + 3b}}{3}$ or $\frac{7 + \sqrt{49 + 42}}{3}$ or $\frac{7 + \sqrt{91}}{3}$ or $\frac{7 + 9, \ldots}{3}$ or 5, ... and e is leſs than 5, ...

But d muſt be leſs than 7, for if $x = 7$ then $7x - x^2 + 14 = 14$ and 7×14 is only 98. Conſequently

the greater root muſt be between 5, ... and 7, and the leaſt root conſequently between 4 and 5, ... Hence, 6 and 5 are naturally the two firſt numbers to be tried, and they both ſucceed.

$$\begin{aligned} x &= 6 \quad 5 \\ \frac{120}{x} &= 20 \quad 24 \\ 7x + 14 - x^2 &= 20 \quad 24 \end{aligned}$$

Let $16x^2 - 31x - x^3 = 264$.

$$\therefore x \times \overline{16x - 31 - x^2} = 264$$

$d + e$ is greater than 16

$16x$ is greater than $31 + x^2$

$$d < \frac{a + \sqrt{a^2 - 3b}}{3} \text{ or } \frac{16 + \sqrt{256 - 93}}{3} \text{ or } \frac{28}{3} \text{ or } 9, \ldots .$$ and e is leſs than 9, ...

Hence, d is between 15, ... and 9, ... and e between 5, ... and 9, ...

$$\begin{aligned} x &= 11 \quad 8 \\ \frac{264}{x} &= 24 \quad 33 \\ 16x - 31 - x^2 &= 24 \quad 33. \end{aligned}$$

Let $513x - 50x^2 - x^3 = 1062$.

$$\therefore x \times \overline{513 - 50x - x^2} = 1062$$

$\therefore$ 513 is greater than $50x + x^2$

$\therefore x$ is leſs than 8, ...

$$d < \frac{\sqrt{a^2 + 3b} - 50}{3} \text{ or } \frac{\sqrt{2500 + 1539} - 50}{3} \text{ or } 63,$$

$\frac{63,\ldots - 50}{3}$ or 4,3 and e is leſs than 4,3

$$x = 6 \quad 3$$
$$\frac{1062}{6} = 177 \quad 354$$
$$513 - 50x - x^2 = 177 \quad 354.$$

5 was not tried as the firſt divider becauſe $\frac{1062}{5}$ would have given a remainder, and $513 - 50x - x^2$ muſt, if x is a whole number, be a whole number.

The expreſſion for the root in equations of the forms admitting two roots when they have only one root may be diſcovered by another method. Since they are ſuppoſed to have only one root, that root muſt be ſuch as to make the unknown ſide the greateſt poſſible; and by adding to, or taking from the root, the unknown ſide is decreaſed. Conſequently, if M is that number which makes the unknown ſide equal to G or the greateſt poſſible, then M $+ y$ or M $- z$ being ſubſtituted for x will make the unknown ſide equal to the ſame number. Let

$$ax^2 + bx - x^3 = k.$$

Then $a\,M^2 + b\,M - M^3 = G$

$$\left.\begin{matrix} a\,x^2 \\ +\,b\,x \\ -\,x^3 \end{matrix}\right\} \left.\begin{matrix} a\,M^2 + 2\,a\,M\,y + a\,y^2 \\ +\,b\,M + b\,y \\ -\,M^3 - 3\,M^2\,y - 3\,M\,y^2 - y^3 \end{matrix}\right\} =$$

$$= \left.\begin{matrix} a\,M^2 - 2\,a\,M\,z + a\,z^2 \\ +\,b\,M - b\,z \\ -\,M^3 + 3\,M^2\,z - 3\,M\,z^2 + z^3 \end{matrix}\right\}$$

$\therefore 2\,a\,M \times \overline{y + z} + b \times \overline{y + z} = 3\,M^2 \times \overline{y + z} + a \times \overline{z^2 - y^2} - 3\,M \times \overline{z^2 - y^2} + z^3 - y^3$

$\therefore$

$$\therefore 2aM + b = 3M^2 + \frac{z^2 - y^2}{z + y} \times a - 3M \times \frac{z^2 - y^2}{z + y} + \frac{z^3 - y^3}{z + y}.$$

Now $\frac{z^2 - y^2}{z + y} \times a - 3M \times \frac{z^2 - y^2}{z + y} + \frac{z^3 - y^3}{z + y}$ muſt be equal to nothing, for z and y may be leſs than any aſſignable number as well as different aſſignable numbers.

$$\therefore 2aM + b = 3M^2$$

$$\therefore M^2 - \frac{2aM}{3} = \frac{b}{3}$$

$$\therefore M = \frac{a + \sqrt{a^2 + 3b}}{3}.$$

In the ſame manner for the other two forms M will be found by making $x = M + y$ and $M - z$, and the reſults from the ſubſtituting of theſe terms for x equal.

Hence, for the equation $ax^2 - bx - x^3 = k$.

$$2aM - 3M^2 = b$$

and for the equation $bx - ax^2 - x^3 = k$

$$b = 2aM + 3M^2.$$

Alſo for the form admitting three roots there will be a number between e and f which makes the unknown ſide greater than any number between e and f can do. Therefore $M + y$, or $M - z$ ſubſtituted for x in the equation, may make the unknown ſide equal to the ſame number.

$$\left.\begin{matrix} x^3 \\ -a\,x^2 \\ +b\,x \end{matrix}\right\}\begin{matrix} \{M^3 + 3\,M^2\,y + 3\,M\,y^2 + y^3 \\ \{-a\,M^2 - 2\,a\,M\,y - a\,y^2 \\ \{+b\,M + b\,y \end{matrix}\Bigg\} =$$

$$= \begin{matrix} \{M^3 - 3\,M^2\,z + 3\,M\,z^2 - z^3 \\ \{-a\,M^2 + 2\,a\,M\,z - a\,z^2 \\ \{+b\,M - b\,z. \end{matrix}$$

$$\therefore 3\,M^2 \times \overline{z+y} = 2\,a\,M \times \overline{z+y} - b \cdot \times \overline{z+y}$$

$$+ 3\,M \times \overline{z^2 - y^2} - a \times \overline{z^2 - y^2} - z^3 + y^3$$

$$\therefore 3\,M^2 = 2\,a\,M - b + \frac{A - B}{z+y}$$

$$\therefore \frac{b}{3} = \frac{2\,a\,M}{3} - M^2$$

$$\therefore M = \frac{a + \sqrt{a^2 - 3\,b}}{3}$$

In this caſe the expreſſion $\frac{a + \sqrt{a^2 - 3\,b}}{3}$ cannot be applied. For $\frac{a + \sqrt{a^2 - 3\,b}}{3}$ is greater than $\frac{a}{3}$. Therefore 2 M would be greater than $\frac{2\,a}{3}$. Conſequently $a - 2\,M$ which is equal to d would be leſs than $\frac{a}{3}$ or the greater root leſs than the ſmaller, which is abſurd. Hence, if the equation has only two roots, the leaſt is equal to $\frac{a - \sqrt{a^2 - 3\,b}}{3}$.

Alſo ſince $a - 2\,M = d$ and $M = \frac{a - \sqrt{a^2 - 3\,b}}{3}$

$$a - \frac{2a - 2\sqrt{a^2 - 3b}}{3} = d = \frac{a + 2\sqrt{a^2 - 3b}}{3}.$$

Hence, if the equation has two roots the greateſt root muſt be equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ and the leaſt muſt be equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$.

Alſo in this form there may be a number between d and e, making the unknown ſide leſs than any other number between d and e can do; and conſequently, if this number is called M, then $M + z$ and $M - y$ ſubſtituted for x may give equal reſults. Hence as before

$$\therefore 3M^2 + b = 2aM$$

$$\therefore \frac{b}{3} = \frac{2aM}{3} - M^2$$

$$\therefore \frac{\sqrt{a^2 - 3b}}{3} = M - \frac{a}{3} \text{ or } \frac{a}{3} - M$$

$$\therefore M = \frac{a \pm \sqrt{a^2 - 3b}}{3}$$

In this caſe $\frac{a - \sqrt{a^2 - 3b}}{3}$ cannot be applied for $a - 2M = f = a - \frac{2a - 2\sqrt{a^2 - 3b}}{3} = \frac{a + 2\sqrt{a^2 - 3b}}{3}$ a number greater than $\frac{a}{3}$, and conſequently

fequently f the leaſt root would be greater than M, which is abſurd. Hence $M = \frac{a + \sqrt{a^2 - 3b}}{3}$.

Hence if the equation has three roots f is leſs than $\frac{a - \sqrt{a^2 - 3b}}{3}$, e is greater than $\frac{a - \sqrt{a^2 - 3b}}{3}$ but leſs than $\frac{a + \sqrt{a^2 - 3b}}{3}$, and d is greater than $\frac{a + \sqrt{a^2 - 3b}}{3}$.

If the equation has two roots, then one is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$ and the other is equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3}$, or one is equal to $\frac{a - \sqrt{a^2 - 3b}}{3}$ and the other is equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$.

If the equation has only one root, then no numbers $M - z$, $M + y$, can make the unknown ſide give equal reſults, and the expreſſions $a \pm \sqrt{a^2 - 3b}$ become impoſſible, or a^2 is either equal to or leſs than $3b$, in either of which caſes $\sqrt{a^2 - 3b}$ cannot expreſs any number whatſoever. But ſince the two roots of the equation $\frac{a + \sqrt{a^2 - 3b}}{3}$, $\frac{a - 2\sqrt{a^2 - 3b}}{3}$ may approach near-

or

er to each other than by any assignable difference, they will approach nearer to $\frac{a}{3}$ than by any assignable difference, and the same may be said of the two roots $\frac{a-\sqrt{a^2-3b}}{3}$, $\frac{a+2\sqrt{a^2-3b}}{3}$, that is, the equation will have only one root when it is equal to $\frac{a}{3}$ and $a^2 = 3b$, or if a^2 is less than $3b$ there can be only one root. There may be also only one root when a^2 is greater than $3b$, in which case the root must be greater than $\frac{a+2\sqrt{a^2-3b}}{3}$ or less than $\frac{a-2\sqrt{a^2-3b}}{3}$.

Instances.

Let $x^3 - 12x^2 + 47x = 60$.

Here $\frac{a-\sqrt{a^2-3b}}{3} = \frac{12-\sqrt{144-141}}{3} = 4 - \frac{\sqrt{3}}{3} = 4 - 0, \ldots, = 3, \ldots$.

and $\frac{a+\sqrt{a^2-3b}}{3} = 4 + \frac{\sqrt{3}}{3} = 4 + 0, \ldots = 4, \ldots$.

Hence,

Hence, if the equation has three roots, one muſt be leſs than 3, ... the next greater than 3, ... but leſs than 4, ... and the third muſt be greater than 4, ...

$$x = 3 \quad 4$$
$$\frac{60}{x} = 20 \quad 15$$
$$x^2 - 12x + 47 = 20 \quad 15.$$

Since 3 and 4 are roots, the third root is found by taking away 3 + 4 from 12. Or the third root is 5.

Let $x^3 - 18x^2 + 105x = 200$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{18 - \sqrt{324 - 315}}{3} = 6 - 1 = 5$$
$$\frac{a + \sqrt{a^2 - 3b}}{3} = 6 + 1 = 7$$
$$x = 5$$
$$\frac{200}{x} = 40$$
$$x^2 - 18x + 105 = 40.$$

Since 5 is a root, the equation can have only two roots and the ſecond root is equal to $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ or $\frac{18 + 6}{3} = 8$.

Let $x^3 - 16x^2 + 77x = 98$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{16 - \sqrt{256 - 231}}{3} = \frac{16 - 5}{3}$$

$$= \frac{11}{3} = 2, \ldots.$$

$$\frac{a + \sqrt{a^2 - 3b}}{3} = \frac{16 + 5}{3} = 7$$

$$x = 7$$

$$\frac{98}{x} = 14$$

$$x^2 - 16x + 77 = 14.$$

Since one root is equal to $\frac{a + \sqrt{a^2 - 3b}}{3}$, the other must be equal to $\frac{a - 2\sqrt{a^2 - 3b}}{3} = \frac{16 - 10}{3}$ or $\frac{6}{3}$ or 2.

Let $x^3 - 12x^2 + 48x = 64$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{12 - \sqrt{144 - 144}}{3} = 4.$$

Here $a^2 = 3b$.

and 4 is therefore the only root.

Let $x^3 - 6x^2 + 20x = 15$.

Here $3b$ is greater than a^2, therefore the equation has only one root, and this root muſt be leſs than 6; for if x is

is made 6 the unknown ſide becomes 120. And if $20x = 6x^2$ or $x = \frac{20}{6} = 3, 3 \ldots$ then x^3 is much greater than 15. Therefore the root muſt be leſs than 3.3. The numbers 2 and 1 therefore naturally preſent themſelves, and 1 is the root.

Let $x^3 - 8x^2 + 20x = 400$.

Here a^2 is greater than $3b$, but the root cannot be leſs than $\frac{a - \sqrt{a^2 - 3b}}{3}$ or $\frac{8 - 2}{3} = 2$, as is evident from inſpection: therefore the equation cannot have three roots. Alſo $\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{8 + 4}{3} =$ which is evidently leſs than the root. Hence, this equation can have only one root, and this root muſt be greater than 8; for if $x = 8$ or $x^3 = 8x^2$ then the unknown ſide $= 20x = 160$, a number leſs than 400.

Since 10 is a divider of 400, it naturally preſents itſelf for trial.

$$x = 10$$

$$\frac{400}{x} = 40$$

$$x^2 - 8x + 20 = 40.$$

Hence 10 is the only root of this equation.

Let $x^3 - 17x^2 + 54x = 350$.

$$\frac{a - \sqrt{a^2 - 3b}}{3} = \frac{17 - \sqrt{289 - 162}}{3} = \frac{17 - \sqrt{127}}{3}$$

$$= \frac{17 - 11, \ldots.}{3} = \frac{5, \ldots.}{3} = 16 \text{ which fails}$$

$$\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{17 + 11, \ldots.}{3} = \frac{28, \ldots.}{3} =$$

9, ... which fails; for if x is 9, the unknown ſide is impoſſible.

$$x = 14\ \ 15$$

$$\frac{350}{x} = 25\ \ 23, \ldots.$$

$$x^2 - 17x + 54 = 12\ \ 24.$$

Hence x is between 14 and 15.

Let $x^3 - 65x^2 + 914x = 98746$.

$$\therefore z^3 - 6{,}5z^2 + 9{,}14z = 98{,}746$$

$$\text{And } z \times \overline{z^2 - 6{,}5z + 9{,}14} = 98{,}746.$$

If $z^3 = 6{,}5z^2$ or $z = 6{,}5$ then $9{,}14z$ will be leſs than 98,746, conſequently z muſt be greater than 6,5.

$$\frac{a + 2\sqrt{a^2 - 3b}}{3} = \frac{6{,}5 + 2\sqrt{42{,}25 - 27{,}42}}{3} =$$

$$\frac{6{,}5 + 2\sqrt{14{,}83}}{3} = \frac{6{,}5 + 6, \ldots.}{3} = \frac{10, \ldots.}{3} = 3, \ldots$$

Conſequently there can be only one root, and that greater from a preceding conſideration than 6,5

$$z = 7 \cdot 7{,}2 \quad 7{,}1$$

$$\frac{98{,}746}{7} = 14{,}\ldots\ 13{,}71\ \ 13{,}9$$

$$z^2 - 6{,}5\,z + 9{,}14 = 12{,}64\ \ 14{,}18\ \ 13{,}40.$$

Consequently $z = 7{,}15 + y$.

Here it may be obſerved that there is not much difficulty in finding the lower numbers; for $z^2 - 6{,}5\,z + 9{,}14$ is reſolvable into $\overline{z \times z - 6{,}5} + 9{,}14$. Conſequently when $z = 7{,}1$, the expreſſion is $7{,}1 \times \overline{7{,}1 - 6{,}5} + 9{,}14$ or $7{,}1 \times {,}6 + 9{,}14 = 4{,}26 + 9{,}14 = 13{,}40$. By means of a book of logarithms the middle row might be continued without difficulty, and thus five or ſix figures might be eaſily found for the root.

In ſearching after any one of the roots in the forms admitting two or three roots, the ſame methods are to be applied as if the equation had only one root. For within the limits, within which the root under examination lies, any addition to x makes the unknown ſide increaſe; any diminution from x makes it decreaſe. Conſequently the reaſoning uſed in forms admitting only one root is applicable to the other forms. Theſe forms are, as was ſaid before

$$x^3 + a\,x^2 + b\,x = k$$
$$x^3 - a\,x^2 - b\,x = k$$
$$x^3 + a\,x^2 - b\,x = k.$$

For the equation $x^3 + a\,x^2 + b\,x = k$ it is evident that x is leſs than $\sqrt[3]{k}$ or than $\sqrt{\frac{k}{a}}$ or than $\frac{k}{b}$, conſequently the leaſt of theſe numbers will be the neareſt limit,

mit, and by ſubſtituting this limit for x another leſs than x may be found.

Let $x^3 + 12x^2 + 42x = 261$.

$\therefore x \times \overline{x^2 + 12x + 42} = 261$

x is leſs than $\sqrt[3]{261}$ or 6, ...

x is leſs than $\sqrt{\frac{261}{12}}$ or $\sqrt{21}$, ... or 4, ...

x is leſs than $\frac{261}{42}$ or $\frac{43,3}{7}$ or 6,19

x	$= 4$	3	
$\frac{261}{4}$	$= 6,5,..87.$	Therefore $x = 3$	
$x^3 + 12x + 42 =$	106	87.	

Let $x^3 + 74x^2 + 8729x = 560783$.

$x = 10z$

$\therefore z^3 + 7,4z^2 + 87,29z = 560,783$

z is leſs than $\sqrt[3]{560.783}$ or 8, ...

z is leſs than $\sqrt{\frac{560,783}{7,4}}$ or 8, ...

z is leſs than $\frac{560,783}{87,29}$ or 6, ...

z	$= 5$	4	4,2	4,148
$\frac{560,783}{z}$	$= 112$	140,195	133,519	135,193
$z \times \overline{z + 7,4} + 87,29 =$	149.29	132,89	136,01	135,191104.

Hence

Hence z is very nearly equal to 4,148 or is equal to $4{,}148 + v$.

Thus $4{,}148 + v$ might be used for the first approach to z, but the method of dividers points out a nearer number.

$$136{,}01 - 133{,}519 = 2{,}491$$
$$135{,}193 - 135{,}191104 = \,,002\ldots$$
$$\text{Sum} = 2{,}493$$

Difference between 4,148 and 4,2 = ,052 and $\frac{2{,}493}{,052}$ = 047 and ,047 × ,05 = ,00234 = ,002 .... nearly

$\therefore z = 4{,}14805$ nearly.

Consequently for the first approach $4{,}14805 \pm v$ might be used, or the process might be continued farther by the mode of dividers, thus:

Increment of $z^2 + 7{,}4z + 87{,}29 = 2z\dot{z} + 7{,}4\dot{z} + \dot{z}^2 =$

$$\dot{z} \times \overline{2z + 7{,}4 + \dot{z}}$$

$z = 4{,}148$ and $\dot{z} = ,00005$.

Increment of $z^2 + 7{,}4z + 8{,}729 =$

$\overline{8{,}296 + 7{,}4 + ,00005} \times ,00005 = ,0007848025$.

To this add 135,191104 and $z^2 + 7{,}4z + 87{,}29$ by this increase from 4,141 to 4,1105 becomes 135,1918888025

$z = 4{,}14805$ 4,148051

$\frac{560{,}783}{4{,}14805} = 135{,}19196$

$z^2 + 7{,}4z + 8{,}729 = 135{,}1918888025$ 135,191900498601.

Hence z is evidently greater than 4,148051, and for the first approach $4{,}148051 + v$ might be used.

This

This number agrees with Raphſon's ſolution, who after three approaches makes the root 4,1480514.

The advantage in uſing dividers is this. The lower ſeries is eaſily found by deriving each term from the preceding by its increment, and when the root does not conſiſt of a great number of figures, the diviſion requiſite for the middle ſeries is not very troubleſome. Alſo the number laſt found in the lower ſeries ſerves in the firſt approach and ſaves a tedious multiplication. Thus if $z = 4{,}148051 + v$ or $a + v$

$$\therefore \overline{4{,}148051 + v} \times$$

$$\overline{a^2 + 7{,}4\,a + 8{,}729 + 2\,a\,v + v^2 + 7{,}4\,v} = 560783.$$

But $a^2 + 7{,}4\,a + 8{,}729$ has already been found to be equal to 135,191900498601

$$\therefore \overline{4{,}148051 + v} \times$$

$$\overline{135{,}191900498601 + 2\,a\,v + v^2 + 7{,}4\,v} = 560{,}783.$$

But by the method of dividers only two figures are gained by each trial, whereas by ſubſtitution when v is very ſmall, the number of figures is frequently more than doubled.

In the firſt inſtance $x^3 + 12\,x^2 + 42\,x = 261$,* x was found to be leſs than 4, ... and as $x = \frac{261}{x^2 + 12\,x + 42}$ if x is leſs than 4, x muſt be greater than $\frac{261}{4^2 + 12 \cdot 4 + 42}$ or $\frac{261}{16 + 48 + 42}$ or $\frac{261}{106}$ or 2, ...

* Page 78.

Hence,

Hence x muſt lie between 2, ... and 4, ...

In general let $x^3 + a x^2 + b x = k$ let d be the leaſt number found which is greater than x. Then $\frac{k}{d^2 + a d + b}$ is leſs than x. Thus the firſt number to be uſed by the mode of dividers is confined within ſufficiently narrow limits.

Let $x^3 - a x^2 - b x = k$.

x^3 is greater than $a x^2 + b x$
$\therefore$ x^2 is greater than $a x + b$.

Let $a x + b = c^2$.

$\therefore$ x^2 is greater than c^2
$\therefore$ x is greater than c.
But $x^3 = k + a x^2 + b x$
$\therefore$ x is greater than $\sqrt{k + a c^2 + b c}$
Alſo $x^3 - k = a x^2 + b x$
$\therefore$ x is greater than $\sqrt[3]{k}$
Alſo x is greater than a and x than $\sqrt{b}$.

Let $x^3 - 15 x^2 - 229 x = 525$.

x is greater than $\sqrt[3]{525}$ or 8, ...
x is greater than 15
x is greater than $\sqrt{229}$ or 15, ...
$x^2 - 15 x$ is greater than 229
$\therefore$ x is greater than 24, ...

$$x = 25$$
$$\frac{525}{x} = 21$$
$$x^2 - 15x - 229 = 21.$$

Let $x^3 - 10x^2 - 91x = 16.$

x is greater than $\sqrt[3]{16}$ or 2, ...
x is greater than 10
x is greater than $\sqrt{91}$ or 9, ...
$x^2 - 10x$ is greater than 91 or 15, ...

x	$= 16$	15,8	15,817
$\frac{16}{x}$	$= 1$	1,0126	1,0115
$x \times \overline{x - 10} - 91$	$= 5$	,64	1,007489.

Hence $x = 15{,}8172 + z.$

Let $x^3 + ax^2 - bx = k.$

Then $x^2 + ax$ is greater than b

$\therefore x$ is greater than $\frac{\sqrt{a^2 + 4b} - a}{2}$

$x^3 = k + bx - ax^2$

If bx is greater than ax^2 or b greater than a then x is greater than $\sqrt[3]{k}$.

But if b is leſs than a then x is leſs than $\sqrt[3]{k}$.

Alſo $ax^2 = k + bx - x^3$.

If b is greater than x^2 then x is greater than $\sqrt{\frac{k}{a}}$.

But

But if b is leſs than x^2, then x is leſs than $\sqrt{\frac{k}{a}}$.

Let $x^3 + 4x^2 - 17x = 12$.

x is greater than $\frac{\sqrt{16+78}-4}{2}$ or 2, ... alſo x is greater than $\sqrt[3]{k}$ or 2, ... and x is greater than $\sqrt{\frac{k}{a}}$ or $\sqrt{\frac{12}{4}}$ or 1, ...

$$x = 3$$
$$\frac{12}{3} = 4 \quad \therefore x = 4$$
$$x^2 + 4x - 17 = 4$$

Let $x^3 + 22x^2 - 103x = 4$.

x is greater than $\sqrt{\left[\frac{a}{2}\right]^2 + b} - \frac{a}{2}$ or $\sqrt{121+103}$ $= 14, \ldots - 11$ or 3, ...

$$x = 4$$
$$\frac{4}{x} = 1 \quad \therefore x = 4$$
$$x^2 + 22x - 103 = 1.$$

Let $x^3 + 6x^2 - 183x = 2704$.

x is greater than $\sqrt{\frac{a^2}{4} + 183} - 3$ or 10, ...

x is greater than $\sqrt[3]{k}$ or $\sqrt[3]{1704}$ or 14, ...

x is leſs than $\sqrt{\frac{k}{a}}$ or $\sqrt{\frac{2704}{6}}$ or 21, ...

$$x \quad = 15 \quad 16$$

$$\frac{2704}{x} \quad = 180 \quad 169$$

$$x^2 + 6x - 183 = 132 \quad 169.$$

Therefore $x = 16$.

ANOTHER mode of diſcovering the number of roots in the equation $x^3 - a x^2 + b x = k$

x^3 may be taken, firſt leſs, then greater than $a x^2$.

If x is taken leſs than a, then the unknown ſide is leſs than $b x$; and if x is made equal to a then the unknown ſide is equal to $b x$. Conſequently, if $b a$ is equal to or leſs than k no number leſs than a can make the unknown ſide equal to k; and ſince the unknown ſide always increaſes, from the time that x is equal to a, by adding to x, the equation can have only one root. Hence if b is equal to or leſs than $\frac{k}{a}$ the equation has only one root.

When x is taken leſs than a,

$$a x^2 - x^3 = b x - k.$$

When x is taken greater than a

$$x^3 - a x^2 = k - b x.$$

In the firſt caſe, namely, when $a x^2 - x^3$ is equal to $b x - k$, let $b x - k = m x$, m being a variable number and

$$a x^2 - x^3 = m x$$

$$\text{or } a x - x^2 = m.$$

Now the ſide $a x - x^2$ firſt increaſes and then decreaſes;

creaſes; but m being equal to $b - \frac{k}{x}$ muſt as x increaſes always increaſe. $a\,x - x^2$ is greateſt when $x = \frac{a}{2}$. Conſequently, if in this caſe $a\,x - x^2$ is leſs than m, no number greater than $\frac{a}{2}$ can be ſubſtituted for x in this equation, and this equation admits only of one root. Hence the given equation $x^3 - a\,x^2 + b\,x = k$ cannot have more than two roots if $\frac{a^2}{4}$ is leſs than m. If $\frac{a^2}{4}$ is greater than m ſome number leſs than $\frac{a}{2}$ will make $a\,x - x^2$ equal to m, and then if the increaſe of $a\,x - x^2$ in that caſe is greater than m, ſome other greater number, but leſs than a, will make $a\,x - x^2$ equal again to m, and thus the equation $a\,x - x^2 = m$ will have two roots, and conſequently the given equation $x^3 - a\,x^2 + b\,x = k$ will have three roots. Now m begins to exiſt at a certain value of x, namely, when $x = \frac{k}{b}$ and its increaſe is greateſt at firſt, but grows continually leſs and leſs. Alſo the increaſe of $a\,x - x^2$ is greateſt at firſt and conſtantly diminiſhes. Hence, m may become equal to $a\,x - x^2$ before x is equal to $\frac{a}{2}$, and then its increaſe may be leſs than that of $a\,x - x^2$, conſequently the equation will have two roots. But if the increaſe of m is greater than that of $a\,x - x^2$, when m firſt becomes equal to $a\,x - x^2$, and is alſo greater when $x = \frac{a}{2}$, then there cannot be another root to the equation $a\,x - x^2 = m$,

m, and conſequently the given equation will have only two roots. A few inſtances will make this ſubject very eaſy to the learner.

Let $x^3 - 18x^2 + 105x = 200$.

When x is leſs than 18, $105x$ may be greater than 200. Conſequently $ax - x^2 = m$.

Let $x = 9$ then $ax - x^2 = 81$ and $m = 105 - 22{,}2 = 82{,}7$. Hence the equation $ax - x^2 = m$ has only one root, and therefore the given equation has only two roots. By trying the expreſſion $\frac{a - \sqrt{a^2 - 3b}}{3}$ or $\frac{18 - \sqrt{324 - 315}}{3}$ or 5 it appears that 5 is a root, and conſequently the other root is $\frac{a + 2\sqrt{a^2 - 3b}}{3}$ or 8.

Let $x^3 - 16x^2 + 77x = 98$.

If x is leſs than 16, $77x$ may be greater than 98, therefore $16x - x^2 = m$.

If $x = 8$, then $16x - x^2 = 64$, and $m = 77 - \frac{98}{8} = 77 - 12{,}25 = 64{,}75$.

Hence, the given equation has only two roots; one of which conſequently is $\frac{a \pm \sqrt{a^2 - 3b}}{3}$ or $\frac{16 \pm \sqrt{256 - 231}}{3}$

or

or $\frac{16 \pm 5}{3}$ or 7 or 3, ... If 7 is one root, then $\frac{16 - 2.5}{3}$ or 2 is the other root, and either 7 or 2 being tried ſucceeds, conſequently the two roots are 7 and 2.

Let $x^3 - 12x^2 + 48x = 64$.

If x is leſs than 12 then $48x$ may be greater than 64, and if $x = 6$ then $12x - x^2 = 36$. But $m = 48 - \frac{64}{6} = 48 - 10{,}66 = 37{,}33$.

Hence it might be preſumed that the equation has two roots; but on applying the expreſſion $\frac{a \pm \sqrt{a^2 - 3b}}{3}$ it appears that $a^2 = 36$, and conſequently this is the limiting caſe in which there is only one root $\frac{a}{3}$ or 4.

Let $x^3 - 6x^2 + 20x = 15$.

If x is leſs than 6, $20x$ may be greater than 15, and if x is made equal to 3, then $6x - x^2 = 9$ and $m = 20 - \frac{15}{3} = 15$.

Hence, the equation has either three roots or only one. If there are three roots one muſt be greater than $\frac{6}{3}$ or 2; but 2 being tried is found to be greater than the root, conſequently the equation can have only one root unity.

Let

Let $x^3 - 8x^2 + 20x = 400$.

If x is leſs than 8, $20x$ cannot be greater than 400, therefore the equation can have only one root, which is ten.

Let $x^3 - 8x^2 + 20x = 13$.

If x is leſs than 8, $20x$ may be greater than 13, and if x is equal to 4, $8x - x^2 = 16$, and $m = 20 - \frac{13}{4} = 20 - 3{,}25 = 16{,}75$. Hence there is a preſumption that the equation has only two roots, which is increaſed by a^2 being greater than $3b$; but 4 and 3 are greater than the root, conſequently the equation has either three roots or one. But if it has three roots, one muſt be greater than 2, ... but any number greater than 2 makes the unknown ſide greater than 13, and conſequently the equation can have only one root unity.

Let $x^3 - 12x^2 + 41x = 42$.

If x is leſs than 12, then $41x$ may be greater than 42, and if $x = 6$ then $12x - x^2 = 36$, and $m = 41 - \frac{42}{6} = 34$. Hence the equation has either three roots or one root; but if it has only one root that root muſt be leſs than $\frac{a - 2\sqrt{a^2 - 3b}}{3}$ or greater than $\frac{a + 2\sqrt{a^2 - 3b}}{3}$, that is, it muſt be leſs than $\frac{12 - 2\sqrt{144 - 123}}{3}$ or leſs than

than unity, which is evidently impoſſible, or it muſt be greater than $\frac{12+2\sqrt{144-123}}{3}$ or greater than 7. On trying the latter ſuppoſition, it appears that 7 is a root, and the equation therefore has two other roots which are eaſily found to be 2 and 3.

Let $x^3 - 15x^2 + 63x = 50$.

If x is leſs than 15 then $63x$ may be greater than 50, and if $x = \frac{15}{2}$, then $15x - x^2 = 56{,}25$ and $m = 63 - \frac{50}{\frac{15}{2}} = 63 - 6{,}66 = 56{,}33$. Hence the equation will have only two roots, or if it has three, two will be very near to each other. $\frac{a \pm \sqrt{a^2-3b}}{3} = \frac{15 \pm \sqrt{225-189}}{3}$ $= \frac{15 \pm 6}{3} = 7$ or 3.

x	$= 3$	7	1
$\frac{50}{x}$	$= 16{,}6$	$7{,}1$	50
$x^2 - 15x + 63$	$= 27$	7	49.

On trying 3 it does not ſucceed; but 7 is very near to the root, conſequently two roots are nearly equal to 7, and the other root muſt be nearly equal to one. On trying unity it appears that it is ſomewhat leſs than the root.

This method of finding the number of roots is improved by comparing together the increaſe of $ax - x^2$ and m, when x is increaſed by a number $\dot{x}$.

By adding $\dot{x}$ to x, $a x - x^2$ becomes

$$a x + a \dot{x} - x^2 - 2 x \dot{x} - \dot{x}^2$$

take away $a x \quad - x^2$

The increaſe is $a \dot{x} - 2 x \dot{x} - \dot{x}^2$ or $\dot{x} \times \overline{a - 2 x - \dot{x}}$

$$m = b - \frac{k}{x}$$

$$m \text{ increaſed} = b - \frac{k}{x + \dot{x}}$$

$\therefore$ increaſe of $m = \frac{k}{x} - \frac{k}{x + \dot{x}} = \frac{k}{x} - \frac{k}{x} + \frac{k \dot{x}}{x^2} - \frac{k \dot{x}^2}{x^3} + \frac{k \dot{x}^3}{x^4} \ldots . \mp \frac{k \dot{x}^{n-2}}{x^{n-1}} \pm \frac{k \dot{x}^{n-1}}{x^n} \times \frac{1}{x + 1} =$

$\frac{k \dot{x}}{x^2} - \frac{k \dot{x}^2}{x^3} + \frac{k \dot{x}^3}{x^4} \ldots \ldots \ldots \mp \frac{k \dot{x}^{n-2}}{x^{n-1}} \pm \frac{k \dot{x}^{n-1}}{x^n}$

$\times \frac{1}{x + 1}$.

And this increaſe, if $\dot{x}$ is very ſmall, is very nearly equal to $\frac{k \dot{x}}{x^2}$.

Conſequently the increaſe of $a x - x^2$ is to the increaſe of m as $a - 2 x$ to $\frac{k}{x^2}$ nearly.

Let $x^3 - 8 x^2 + 20 x = 13$.

Here $m = 20 - \frac{13}{x}$ and begins from $\frac{13}{20}$, that is, m cannot be leſs than or equal to $\frac{13}{20}$. Hence unity is taken as the firſt value of x in comparing $a x - x^2$ with m, or $8 x - x^2$ with $20 - \frac{13}{x}$, and it appears that if $x = 1$ then

$8x - x^2 = 20 - \frac{13}{x} = 7$, conſequently unity is a root of the propoſed equation; and here the increaſe of m is greater than that of $ax - x^2$. If $x = 1$, the increaſe of $m = 6,5$ and that of $ax - x^2 = 5$. If x again is equal to unity, ſince x is now equal to two, the increaſe of $ax - x^2$ is greater than that of m, but m is greater than $ax - x^2$ when $x = 2$, and is alſo greater than $ax - x^2$ when $x = 3$, but they are ſo nearly equal to each other in this laſt caſe, and the increaſe of $ax - x^2$ is ſo much greater than m that the trouble of comparing the farther proceſs of the increaſe of each is greater than that of trying the equation by means of the diſcovered root unity. From this root, by means of the equation

$$x^2 + ex + e^2 + b = a \times \overline{x + e}$$

$$\text{or } x^2 + x + 1 + 20 = 8 \times \overline{x + 1}$$

$$\text{or } 13 = 7x - x^2$$

it appears, that there cannot be another root, for $\left|\frac{7}{2}\right|^2$ is leſs than 13. But as $\left|\frac{7}{2}\right|^2$ is nearly equal to 13, the reaſon of the above proceſs being troubleſome is apparent.

Let $x^3 - 18x^2 + 105x = 200$.

Here the limit of m is $\frac{200}{105}$ or 1, ... and when $x = 2$, $m = 5$ and $18x - x^2 = 32$; but when $x = 2$, the increaſe of m is to that of $ax - x^2$ as 50 to 14, and when $x = 5$, the increaſe of m is equal to that of $ax - x^2$. Conſequently $ax - x^2$ is equal to m when x is either

 equal

equal to five or nearly equal to five. On trying five, it appears that five is a root; but as the increaſe of m is again equal to that of $a\,x - x^2$, when x is ſome number between 5 and 9, there will be another root to the equation, and conſequently the given equation, ſince it can have no root when x is equal to or greater than 18, can have but two roots.

Let $x^3 - 16\,x^2 + 77\,x = 98$.

The limit of m is $\frac{98}{77}$ or 1, ... Conſequently 2 is tried for x, and it appears that $16\,x - x^2$ is then equal to m or $77 - \frac{98}{2}$. The increaſe of $16\,x - x^2$ is firſt leſs than that of m, and then greater and then leſs again, and they are equal when $x = 7$. Conſequently the other root will be either 7 or nearly equal to 7, and as 7 ſucceeds there can be no other root.

One advantage attending this mode is, that it gives a learner a practical knowledge of equations, which will be very uſeful as he advances in ſcience.

Equations of this form may be changed into equations of the ſecond order, and thus admit frequently an eaſy mode of ſolution *.

$$x^3 - a\,x^2 + b\,x = k.$$

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* See Part I. page 139.

Let

Let $x = \frac{a}{3} \pm z$.

$$\left.\begin{array}{l} \therefore x^3 \left\{ \begin{array}{l} \left(\frac{a}{3}\right)^3 \pm 3 \cdot \left(\frac{a}{3}\right)^2 z + 3 \frac{a}{3} z^2 \pm z^3 \\ -a \cdot \left(\frac{a}{3}\right)^2 \mp 2 \cdot \frac{aa}{3} \cdot z - a z^2 \\ b \cdot \frac{a}{3} + b \cdot z \end{array} \right. \\ = a x^2 \\ + b x \end{array} \right\} = k.$$

In the new equation, one term is $z^2 \times \frac{3a}{3} - a$, that is $z^2 \times \overline{a - a}$ or o. Hence, in any propoſed equation of this form, by making x equal to $\frac{a}{3} \pm z$, a new equation of the ſecond order is the reſult, in which there is not an unknown number raiſed to the ſecond power.

Inſtance.

$$x^3 - 3x^2 - 36x = 342.$$

Let $x = \frac{a}{3} + y = 1 + y$

$$\left.\begin{array}{l} \therefore x^3 = 1 + 3y + 3y^2 + y^3 \\ -3x^2 = -3 - 6y - 3y^2 \\ -36x = -36 - 36y \end{array} \right\} = 342$$

$$\therefore -38 - 39y + y^3 = 342$$

$$\therefore y^3 - 39y = 380$$

If $y = 8 \quad 9$

$$\frac{380}{y} = 47{,}5 \quad 42{,}22$$

$$y^2 - 39 = 25 \quad 42$$

$\therefore y$

$$\therefore y = 9 + z$$

$$\left.\begin{array}{rl} y^3 & 729 + 243\,z + 27\,z^2 + z^3 \\ -39\,y & -351 - 39\,z \end{array}\right\} = 380$$

$$\therefore 378 + 204\,z + 27\,z^2 + z^3 = 380$$

$$\therefore 204\,z = 2 - 27\,z^2 - z^3$$

$$\therefore 68\,z = ,66\beta - 9\,z^2 - \frac{z^3}{3}$$

$\therefore z = ,0097912$ and $y = 9 + z = 9,0097912$

But $x = 1 + y \quad \therefore x = 10,0097912$

$$\therefore x^3 = 1002,940236966582020054528$$

$$\left.\begin{array}{rl} -3\,x^2 = & -300,587759602792 32 \\ -36\,x = & -360,3524832 \end{array}\right\} = -$$

$$\text{Sum} = 660,940242802792 32$$

$$\therefore x^3 - 3x^2 - 36\,x = 341,999994163789700 54528$$

which is less than 342 by ,00000 583, &c.

The learner will see the advantage of this change by solving the original equation $x^3 - 3\,x^2 - 36\,x = 342$ in its present state, and having discovered that x is nearly equal to ten, he will make it equal to $10 + z$, and then proceed by the usual mode of approach.

By changing the equation in this manner the root may be discovered by a comparison of the dividers of the known terms in the original and the new equation *.

Let $x^3 + 3\,x^2 + 2\,x = 24$.

x is a divider of 24, and the dividers of 24 are 1 2 3 4 6 8 12 24.

* See Part I. page 140.

The

The new equation is $y^3 - y = 24$.

y is therefore a divider of 24, and $y = x + 1$. Therefore, by adding unity to one of the dividers of 24, a number will be found which is alſo a divider of 24, and conſequently may be a root of the propoſed equation. On examination there are three numbers anſwering to this condition 1 2 3. Conſequently one of them muſt be the root of the propoſed equation; but if $x = 3$ then x^3 is alone greater than 24, and 3 cannot be a root. Unity alſo cannot evidently be the root. Therefore 2 is the root of the given equation.

Let $x^3 - 18x^2 + 105x = 200$.

Dividers 1 2 4 5 8 10.

Let $x = 6 \pm y$.

The new known term is either $200 - 6^3 + 18 \cdot 6^2 - 105 \cdot 6$, which is equal to 2, whoſe dividers are 1 and 2.

Hence 4 5 8 anſwer to the conditions, and they cannot all three be roots, for $4 + 5 + 8$ is not equal to 18, and 8 and 5 being roots there cannot be another root, for 6, which muſt be the root, if the equation has three roots, is not a divider of 200.

If x had been made equal to $y - 6$ the new known term would be $200 + 6^3 + 18 \cdot 6^2 + 105 \cdot 6$ or 1694, whoſe dividers are 1 2 7 11 14, and conſequently x is to be looked for among the numbers $7 - 6$, $11 - 6$, $14 - 6$ or 1, 5 and 8, and, as before, the two roots are found to be 5 and 8.

Let

Let $x^3 - 12x^2 + 48x = 64$.

The dividers of 64 are 1 2 4 8, the new known term is $64 \mp 64 + 12 \times 16 \mp 48 \times 4$ by making $x = 4 \pm y$, that is 0 or 512. Hence it is evident that $x = 4$.

This mode of ſolution can ſeldom be applied with advantage unleſs the copart of the ſecond term $a x^2$ is diviſible by three.

The rule of double poſition for the ſolution of equations is recommended by ſuch high authority that it deſerves particular examination, and to do this with greater juſtice, the rule ſhall be given in the recommender's words, together with his own example, to be compared with the ſame equation ſolved by the method of dividers.

"RULE.

"1. FIND, by trial, two numbers, as near the true root as poſſible, and ſubſtitute them in the given equation inſtead of the unknown quantity; marking the errours which ariſe from each of them.

"2. Multiply the difference of the two numbers, found by trial, by the leaſt errour, and divide the product by the difference of the errours, when they are alike, but by their ſum when they are unlike. Or ſay, As the difference or ſum of the errours is to the difference of the two numbers, ſo is the leaſt errour to the correction of its ſuppoſed number.

"3. Add the quotient, laſt found, to the number belonging

ing to the leaſt errour, when that number is too little, but ſubtract it when too great, and the reſult will give the true root *nearly*.

"4. Take this root and the neareſt of the two former, or any other that may be found nearer; and by proceeding in like manner, a root will be had ſtill nearer than before; and ſo on to any degree of exactneſs required.

"EXAMPLE.

"To find the root of the equation $x^3 + x^2 + x = 100$, or the value of x in it.

"Here it is ſoon found that x lies between 4 and 5. Aſſume therefore theſe two numbers, and the operation will be as follows:

1ſt Sup.		2d Sup.
4	x	5
16	x^2	25
64	x^3	125
84	ſums	155
−16	errours	+55

the ſum of which is 71.
Then as 71 : 1 :: 16 : ·225.
Hence $x = 4{\cdot}225$ nearly.

"Again, ſuppoſe 4·2 and 4·3, and repeat the work as follows:

1ſt Sup.		2d Sup.
4·2	x	4·3
17·64	x^2	18·49
74·088	x^3	79·507
95·928	ſums	102·297
−4·072	errours	+2·297

the ſum of which is 6·369
As 6·369 : 1 :: 2·297 : 0·036
This taken from 4·300
leaves x nearly = 4·264

" Again, ſuppoſe 4·264 and 4·265, and work as follows:

4·264	-	x	-	4·265
18·181696	-	x^2	-	18·190225
77·526752	-	x^3	-	77·581310
99·972448	-	ſums	-	100·036535
−0·027552	-	errours	-	+0·036535

the ſum of which is ·064087.

Then as ·064087 : ·001 :: 4·264 : 0·0004299
To this adding - 4·264
gives x very nearly = 4·2644299."

The ſame example worked out by the mode of dividers.

Let $x^3 + x^2 + x = 100$.

Therefore $x \times \overline{x^2 + x + 1} = 100$

If x	=	4	4,26	4,27
$\frac{100}{x}$	=	25 20	23,4741	23,4192
$x^2 + x + 1$	=	21 31	23,4076	23,5029
Diff.	=	4 11	,0665	,0836

$4 + 11 = 15$ $\frac{15}{10} = 1{,}5$ and $\frac{4}{1{,}5} = \frac{,8}{,3} = \frac{,266}{,1}$

Hence 4,26 muſt be a nearer value of x, and on trying 4,26 it appears to be ſo near to x that 4,27 muſt be greater than x.

,0665

$\frac{,0665 + ,0836}{10} = \frac{,1501}{10} = ,01501$ and $\frac{,06650}{,01501} = 4,430$.

Hence, x is equal to 4,26443 nearly.

By the mode of double poſition, the firſt ſtep gives 4,225, and to do this the diviſion of 16 by 71 to four places is neceſſary.

By the mode of dividers the firſt ſtep gives a nearer number for the root, namely, 4,26, and a diviſion by one figure only is neceſſary.

By the mode of double poſition, the ſecond ſtep gives 4,264; but to do this the numbers 4,2 4,3 muſt be raiſed to the third powers, and 2,297 muſt be divided by 6,369 as far as four places.

By the mode of dividers, the ſecond ſtep gives 4,26443, and to do this 100 muſt be divided by each of the numbers 426 and 427 to ſix places, and the numbers 4,26 and 4,27 muſt be raiſed to the ſecond powers or derived by the mode of increments more eaſily from the value of $x^2 + x + 1$ when $x = 4$.

By the mode of double poſition 4,264 and 4,265 muſt be raiſed to the third powers, and 4,264 is to be divided by ,064087; and when this operation is performed the reſult is very nearly the ſame with that obtained by the mode of dividers by two operations, each eaſier than the correſponding operation by the rule of double poſition.

Thus, by double position $x = 4{,}2644299$
by dividers - $x = 4{,}2644303$
difference - - $= 0{,}0000004$

Hence it appears, that the mode of dividers is far preferable to that by double position, and the number 4,26443 having been obtained x should be made equal to 4,26443 $\pm z$, and by the method of approach a greater number of figures will be obtained than by either the rule of double position or the mode of dividers.

Another mode stands recommended by the authority of Dr. Halley, but it is probable that very few persons since his time have adopted it, and a very accurate investigation of it by Baron Maseres in several instances has proved its inferiority to the simple method of approach. Instead of rejecting all the powers of z in the new equation, as in the simple mode of approach, Dr. Halley retains the terms, having in them z^2. Thus let $x^3 - a x^2 + b x = c$.

$$\text{Make } x = d \pm z$$

$$\left.\begin{array}{l} \therefore x^3 \\ -a x^2 \\ +b x \end{array}\right\} \begin{array}{l} d^3 \pm 3 d^2 z + 3 d z^2 \pm z^3 \\ -a d^2 \mp 2 a d z - a z^2 \\ +b d \pm b z \end{array} \Bigg\} = k$$

By neglecting only the term z^3 there remains an equation of the second class to be solved, the copart of whose simple term z being $\frac{\pm 3 d^2 \mp 2 a d \pm b}{3 d - a}$, will, after the first operation, be a fraction with a considerable number of figures in both the upper and the lower parts.

Now

Now half of this fraction is to be raiſed to the ſecond power, and then the ſecond root of the known term added to the new fraction is to be extracted, ſo that though a greater number of figures will be attained by each operation, yet the trouble of obtaining them far outbalances this advantage, and in practice it will be found much eaſier to uſe more operations by the ſimple mode of approach to arrive at the ſame number of figures.

CHAP. V.

EQUATIONS OF THE FOURTH CLASS.

EQUATIONS of the fourth claſs have four terms on the unknown ſide, and are divided into four orders. Thoſe in the firſt order can have only one root, in the ſecond only two, in the third three roots, and in the fourth four roots.

I.
$$\begin{cases} 1.\ x^m + a\,x^n + b\,x^o + c\,x^p = k \\ 2.\ x^m + a\,x^n + b\,x^o - c\,x^p = k \\ 3.\ x^m + a\,x^n - b\,x^o - c\,x^p = k \\ 4.\ x^m - a\,x^n - b\,x^o - c\,x^p = k. \end{cases}$$

II.
$$\begin{cases} 1.\ a\,x^n + b\,x^o + c\,x^p - x^m = k \\ 2.\ a\,x^n + b\,x^o - c\,x^p - x^m = k \\ 3.\ a\,x^n - b\,x^o - c\,x^p - x^m = k \\ 4.\ b\,x^o + c\,x^p - a\,x^n - x^m = k \\ 5.\ b\,x^o - c\,x^p - a\,x^n - x^m = k \\ 6.\ c\,x^p - b\,x^o - a\,x^n - x^m = k. \end{cases}$$

III.

III.
$$\begin{cases} 1.\ x^m - a x^n + b x^o + c x^p = k \\ 2.\ x^m + a x^n - b x^o + c x^p = k \\ 3.\ x^m - a x^n - b x^o + c x^p = k \\ 4.\ x^m - a x^n + b x^o - c x^p = k. \end{cases}$$

IV. $$c x^p - b x^o + a x^n - x^m = k.$$

In equations of the firſt and third orders the terms on the unknown ſide are written down according to the indexes of the unknown powers $m\ n\ o\ p$. In the ſecond and fourth orders the terms are not written down in the ſucceſſion of the indexes, but the higheſt index is in the laſt term. In the firſt order there is either no change or only one change of the marks of adding and ſubtracting. In the third order there are either two or three changes of the marks of adding and ſubtracting. Hence, if an equation in theſe orders is written down agreeably to the indexes, the number of roots in the form to which it belongs is determined by obſerving the change in the marks: if there is either no change or one change the equation can have only one root: if there are either two or three changes (that is, if there is more than one change) the equation belongs to a form capable of having three roots.

In the ſecond order there are either one or two changes of the marks; in the fourth order there are three changes of the marks. Theſe changes are obſerved by reading the equation according to the indexes $m\ n\ o\ p$. Thus, in the equation

$$b x^o - c x^p - a x^n - x^m = k$$

by reading it in the order preſcribed $- x^m$, $- a x^n$, $+ b x^o$, $- c x^p$ the two firſt terms x^m and $a x^n$ have the ſame

ſame mark, the third term has a different mark, conſequently there is one change in reading the three terms x^m, $a x^n$, $b x^o$, and the two terms $b x^o$, $c x^p$ have different marks, conſequently there are two changes of marks by reading according to the order of the indexes. Hence, if in an equation of this claſs, the unknown term, with the higheſt index, is not the firſt term, the equation belongs to the ſecond or the fourth order. If there are either one or two changes of the marks, the equation is of the form capable of having two roots. If there are three changes of the marks, the equation is of the fourth order, and of a form capable of having four roots.

Let it be required to determine the number of roots in the equation $54 x^3 - 32 x^2 + 20 x^4 - 50 x = 200$.

The equation written in order is $20 x^4 + 54 x^3 - 32 x^2 - 50 x = 200$, conſequently the equation having only one change, and the higheſt power of the unknown number being in the firſt term, the equation can have only one root.

Let $30 x^2 - 12 x - 50 x^4 + 172 x^3 = 340$.

$\therefore 172 x^3 + 30 x^2 - 12 x - 50 x^4 = 340$.

The term x^4 being laſt, and there being two changes in the marks, the equation cannot have more than two roots.

Let $20 x - 50 x^3 - 30 x^3 + 40 x^4 = 2012$.

$\therefore 40 x^4 - 30 x^3 - 50 x^2 + 20 x = 2012$.

The term $40 x^4$ being the firſt, and there being two changes

changes of the marks, the equation cannot have more than three roots.

Let $70x^3 - 12x^2 + 50x - 30x^4 = 59$.

$\therefore 50x - 12x^2 + 70x^3 - 30x^4 = 59$.

The term $30x^4$ being the laſt, and there being three changes of the marks, the equation is of the fourth order, and conſequently is of a form capable of having four roots.

To find the relation of the coparts and roots in the equation

$$cx - bx^2 + ax^3 - x^4 = k.$$

Let e, f, g, h be the roots. Then

$$ce - be^2 + ae^3 - e^4 = k$$

and $cf - bf^2 + af^3 - f^4 = k$

$\therefore c \times \overline{e-f} - b \times \overline{e^2-f^2} + a \times \overline{e^3-f^3} = e^4 - f^4$

$\therefore c - b \times \overline{e+f} + a \times \overline{e^2+ef+f^2} = e^3 + e^2f + ef^2 + f^3$

$c - b \times \overline{e+g} + a \times e^2 + eg + g^2 = e^3 + e^2g + eg^2 + g^3$

$\therefore ae \times \overline{f-g} + a \times \overline{f^2-g^2} - b \times \overline{f-g} = e^2 \times \overline{f-g} + e \times \overline{f^2-g^2} + f^3 - g^3$

$\therefore ae + a \times \overline{f+g} - b = e^2 + e \times \overline{f+g} + f^2 + fg + g^2$

$ae + a \times \overline{f+h} - b = e^2 + e \times \overline{f+h} + f^2 + fh + h^2$

$\therefore a \times \overline{g-h} = e \times \overline{g-h} + f \times \overline{g-h} + g^2 - h^2$

$$\therefore a = e + f + g + h.$$

By the ſame proceſs as in pages 25, 52, it is proved that b is equal to the ſum of the products of each pair of roots,

roots, c the ſum of the products of the roots taken by three's, k the product of all the roots.

If one of the roots of an equation in this order is known, the other three may be found by means of an equation of the third claſs. Let e be the root known, then the equation to find the roots is

$$e - b \times \overline{e + x} + a \times \overline{e^2 + ex + x^2} = e^3 + e^2x + ex^2 + x^3.$$

Inſtance.

$$11208\,x - 1708\,x^2 + 78\,x^3 - x^4 = 16192$$
$$x = 1 \quad 2$$
$$\frac{16192}{x} = 16192 \quad 8096$$
$$11208 - 1708x + 78\,x^2 - x^3 = 9577 \quad 8096.$$

Hence 2 is a root of the propoſed equation.

$$\therefore 11208 - 1708 \times \overline{2 + x} + 78 \times \overline{4 + 2x + x^2} = 8 + 4x + 2x^2 + x^3$$
$$\therefore 11208 - 3416 + 312 - 8 = 1708\,x - 156\,x + 4x - 78\,x^2 + 2\,x^2 + x^3$$
$$\therefore 8096 = 1556\,x - 76\,x^2 + x^3$$
$$x = 10 \quad 8$$
$$\frac{8096}{x} = 809{,}6 \quad 1012$$
$$1556 - 76\,x + x^2 = 896 \quad 1012.$$

Hence 8 is another root of the propoſed equation, and *

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* See page 53.

$$x^2 + 8x + 64 + 1556 = 76 \times \overline{8 + x}$$
$$1620 - 608 = 76x - 8x - x^2$$
$$\therefore 1012 = 68x - x^2$$
$$\therefore \sqrt{34^2 - 1012} = \begin{cases} 34 - x \text{ or} \\ x - 34 \end{cases}$$
$$\therefore 12 = \begin{cases} 34 - x \text{ or} \\ x - 34 \end{cases}$$
$$\therefore x = 46 \text{ or } 22.$$

Hence the roots of the propoſed equation are 2, 8, 22, 46 and

$$2 + 8 + 22 + 46 = 78.$$

Alſo $2 \times 8 + 2 \times 22 + 2.46 + 8 \times 22 + 8 \times 46 + 22 \times 46 = 1708$

And $2 \times 8 \times 22 + 2 \times 22 \times 46 + 2 \times 8 \times 46 + 8 \times 22 \times 46 = 11208.$

Alſo $2 \times 8 \times 22 \times 46 = 16192.$

If the equation is of either of the orders admitting more than one root, and one root is known, the remaining root or roots may be diſcovered by an equation of the third claſs.

The relations between the coparts and the roots of theſe equations may be found in the ſame manner as in the equations of the lower claſſes.

When the equation has more than one root, ſome number between any two roots being ſubſtituted for x, will make the unknown ſide greater than any other between thoſe two roots can do; and to find this, make $x = M + y$ or $M - z$ as in pages 29—67: and, by comparing together

together in the ſame manner the two equations thus formed, an equation will be diſcovered, whoſe roots are the different values of M, or the limits between the roots of the propoſed equation. Thus let the equation be of the form

$$x^4 - a x^3 + b x^2 - c x = k$$

The limiting equation is

$$4 x^3 - 3 a x^2 + 2 b x = c$$

or each term is to be multiplied into the index of the unknown term in it, and the marks of all the unknown terms are to remain the ſame except that of the laſt unknown term, which is made equal to the ſum or difference of the other unknown terms thus found, and then the new equation is divided by x. Let

$$c x - b x^2 - a x^3 - x^4 = k.$$

Then for the limiting equation

$$c x - 2 b x^2 - 3 a x^3 = 4 x^4$$

$$\therefore c - 2 b x - 3 a x^2 = 4 x^3.$$

Inſtance.

Let $11208 x - 1708 x^2 + 78 x^3 - x^4 = 16192$.

The limiting equation is

$$11208 x - 2.1708 x^2 + 3.78 x^3 = 4 x^4$$

or

$$11208 - 2.1708 x + 3.78 x^2 = 4 x^3$$

$$\therefore 2802 = 854 x - 58{,}5 x^2 + x^3$$

x	$= 5$	4	10	15	16	
$\frac{2802}{x}$	$= 560{,}4$	700	280,2	186	175	
$854 - 58{,}5 x + x^2$	$= 586{,}5$	636	369	202,5	174.	

 Hence

Hence the three roots of the limiting equation are 4, ... 16, ... and 37, ... and conſequently the roots of the propoſed equation are the leaſt, leſs than four, the next between 4, ... and 16, ... the third between 16, ... and 37, ... and the fourth between 37, ... and 57. The laſt number 57 is found by taking 4, ... + 16, ... from 78; for if two of the roots are greater than 4, ... + 16, ... that is than 20, ... the other two roots muſt be leſs than 78 — 20, ... that is than 57, ... and conſequently the greateſt root muſt be leſs than 57. In this caſe the limiting equation is of little or rather of no uſe towards the diſcovery of the roots of the propoſed equation, as the four roots of the propoſed equation are diſcovered with as little difficulty as theſe limiting roots.

The modes adopted for diſcovering the relation between the coparts and roots of equations of the ſecond and third claſſes may be applied to this claſs, as may alſo the various methods for determining the number of roots in any order of this claſs. But from the greater number of unknown terms the trouble is rather increaſed.

Inſtances.

$$x^4 - 20x^3 + 9x^2 + 100x = 9246.$$

If x is leſs than 20, the unknown term is leſs than $9x^2 + 100x$, and if x is equal to 20 the unknown term is leſs than 9246, conſequently the equation has only one root; for by adding to x, when the unknown term is equal to 9246,

9246, it is increaſed, and by taking away from x the unknown term is diminiſhed.

$$14937\,x - 1998\,x^2 + 80\,x^3 - x^4 = 5000.$$

One root is evidently leſs than unity, and the unknown ſide may be divided into two parts, of which the one $14937\,x - 1998\,x^2$ increaſes till x is equal to $\frac{14937}{2.1998}$ or 3, ... and then decreaſes till x is equal to $\frac{14937}{1998}$ or 7, ... The other part $80\,x^3 - x^4$ increaſes till x is equal to 60, and then decreaſes. Conſequently, ſince one root is leſs than unity there cannot be another root leſs than 3, ... Nor can there be another root between 3, ... and 7, ... for if $x = 4$, then $80\,x^3 - x^4$ is greater than 5000, and conſequently the unknown ſide muſt be greater than 5000 when x is between 3, ... and 7, ... When x is greater than 7, ... then the part $14937\,x - 1998\,x^2$ is not to be added to $80\,x^3 - x^4$, but $1998\,x^2 - 14937\,x$ is to be taken away from it, and the increaſe of $1998\,x^2 - 14937\,x$ is when x is 8 or a number greater than 8 conſiderably greater than the increaſe of $80\,x^3 - x^4$. Conſequently there will be another root to this equation; but when $x = 20$ the increaſe of $80\,x^3 - x^4$ is nearly equal to that of $1998\,x^2 - 14937\,x$, and when x is 30 the increaſe of $80\,x^3 - x^4$ is much greater than that of $1998\,x^2 - 14937\,x$, conſequently there will be a root between 7, ... and 20. Now the part $1998\,x^2 - 14937\,x$ conſtantly increaſes; but there is no increaſe of $80\,x^3 - x^4$ when $x = 60$, conſequently when x is between 30 and 60 the unknown ſide muſt become again equal to 5000,

5000, and unleſs this takes place when the unknown ſide is the greateſt poſſible there muſt be two more roots to the equation *. But it is evident that the number making the unknown ſide the greateſt poſſible is fractional, and therefore there will be four roots to the equation.

Call $14937 - 1998\,x + 80\,x^2 - x^3$ equal to z. Then $x \times z = 5000$

$$x = 10 \quad 12 \quad 13 \quad 30$$
$$\frac{5000}{x} = 500 \quad 416 \quad 384 \quad 166$$
$$z = 1957 \quad 753 \quad 286 \quad 3$$
$$x = 32 \quad 34 \quad 35$$
$$\frac{5000}{x} = 156 \quad 147 \quad 142$$
$$z = 153 \quad 181 \quad 132.$$

Hence the four roots are the one leſs than unity, the ſecond between 12 and 13, the third between 32 and 33, but very nearly equal to 32, the fourth between 34 and 35, but much nearer to the latter than the former number.

Dr. Wallis and Mr. Raphſon have employed themſelves upon this equation; the former giving for one root 12,7564417944807440 2 . . . . the latter in his nine-

* The equation for finding the numbers which make the unknown ſide the greateſt is

$$14937 = 3996\,x - 240\,x^2 + 4x^3.$$

In this equation x cannot be a whole number, for whether x is made an odd or an even number the unknown ſide is an even number, and conſequently not equal to 14937.

teenth

teenth problem making this root 12,756441794481 11, and Baron Maſeres, who is now employed upon the ſame equation, confirms * the ſolution given by Dr. Wallis.

The progreſs in approaching to theſe roots is as follows: The firſt root is evidently leſs than unity. 10 is both the eaſieſt and moſt obvious number to be tried for the next root which led to the trial of 12 and 13. Hence, two of the roots were found to be the one leſs than unity, and the other between 12 and 13. Conſequently the ſum of the remaining roots being equal to the difference between 80 and the ſum of the roots thus found is 67, ... nearly, the half of which being 33, ... is greater than one and leſs than the other remaining root. Hence 30 was tried, which led to the trial of 32, and 32 being found to be very nearly equal to one root, the other is neceſſarily very nearly equal to 34.

$$x^5 - 7x^4 + 20x^3 - 155x^2 = 10000.$$

The unknown ſide of this equation may be divided into two parts $x^5 - 7x^4$ and $20x^3 - 155x^2$. If x is leſs than 7, the unknown ſide is leſs than $20x^3 - 155x^2$. Alſo ſince $20x^3 - 155x^2 = x^2 \times \overline{20x - 155}$, if x is leſs than $\frac{155}{20}$ or 7,75, the unknown ſide is leſs than $x^5 - 7x^4$; and if when $x = 7{,}75$, the term $x^5 - 7x^4$ is leſs than 10000, the equation can have only one root, for when x is greater than 7,75 both parts $x^5 - 7x^4$ and

* In an Appendix, now in the preſs, to Halley's new, exact, and eaſie Method of finding the Roots of any Equations generally.

$20\,x^3 - 155\,x$ increaſe by adding $\dot{x}$ to x, and decreaſe by taking $\dot{x}$ from x. Let $x = 7{,}75$, then the unknown ſide is equal to $x^4 \times \overline{x - 7}$ or $x^4 \times \overline{7{,}75 - 7}$ or $,75 \times \overline{7{,}75}\,|^4$, that is, it is leſs than $,75 \times 8^4$ or $,75 \times 4096$, and conſequently is leſs than 10000. Therefore this equation has only one root.

The ſame thing may be proved by the limiting equation, which is

$$5\,x^4 - 28\,x^3 + 60\,x^2 = 310$$
$$\text{or } x^4 - 5{,}6\,x^3 + 12\,x^2 = 62.$$

It is evident, that if x is equal to or greater than 5,6, the unknown ſide is greater than 62, conſequently the roots of this equation muſt, if there are more than one, be each leſs than 5,6. But 5,6, or a number leſs than 5,6 cannot be a root of the equation.

$$x^5 - 7\,x^4 + 20\,x^3 - 155\,x^2 = 10000$$

which can therefore have but one root.

This equation is ſolved by Baron Maſeres in the third volume of the Scriptores Logarithmici, page 742, and the root determined to be very nearly equal to 8,6 or 8,5992325.

CHAP.

CHAP. VI.

GENERAL OBSERVATIONS.

BY the method purſued in the preceding chapters the relation between the coparts and the roots of any equation whatſoever will be diſcovered, and the equations of one mode in the laſt order of each claſs follow an analogy, which may eaſily be traced from the ſecond order of equations of the ſecond claſs to the laſt order of every claſs. This mode is when the difference between every two adjoining indexes is unity, and the higheſt index is the number which denominates the claſs to which the equation belongs. Thus $a x^n - x^m = k$ is the ſecond order of equations of the ſecond claſs; let $m = 2$ and $n = 1$, then a is equal to the ſum of the roots and k is equal to the product of the roots. $x^3 - a x^2 + b x = k$ is of the third order of the third claſs; the higheſt index is the number denoting the claſs, and the adjoining indexes differ by unity. a is equal to the ſum of the roots; b is equal to the ſum of the product of each pair of roots, and k is equal to the product of the roots. $c x - b x^2 + a x^3 - x^4 = k$. Here the equation is of the laſt order of the fourth claſs, the higheſt index is the number denoting the claſs, and the adjoining indexes differ by unity. a is equal to the ſum of the roots; b the ſum of the products of each pair of roots; c the ſum of the products of the roots taken by threes, and k the product of all the roots. Let n denote the higheſt index, and let the adjoining indexes differ by unity, then the equations of the laſt order of a claſs belong to one or

other of theſe forms according as the claſs is denoted by an even or an odd number

$$l x - \ldots + \ldots - \ldots + \ldots - d x^{n-4} + c x^{n-3} - b x^{n-2} + a x^{n-1} - x^n = k$$

$$x^n - a x^{n-1} + b x^{n-2} - c x^{n-3} + d x^{n-4} - e x^{n-5} \ldots = k.$$

In both caſes, if the equation has n roots, a is equal to the ſum of the roots, b is equal to the ſum of the products of each pair of roots, c is equal to the ſum of the products of the roots taken by threes, d is equal to the ſum of the products of the roots taken by fours, and ſo on, and k is the product of all the roots.

The following is a general form to which every propoſed equation may be made to correſpond, obſerving only that no equation whatever can have all the loweſt marks together:

$$\pm x^m \pm a x^n \pm b x^o \pm c x^p \pm d x^q \pm e x^r \ldots = k.$$

The limiting equation to the above equation is

$$\pm m x^{m-1} \pm n a x^{n-1} \pm o b x^{o-1} \pm p c x^{p-1} \ldots = \pm b x^{t-1} \pm v x^{v-1}.$$

The proof is the ſame as that adopted in pages 26, 27, 29, 67, 69.

It has been already obſerved, that this limiting equation will be of little uſe in equations of high claſſes. As an inſtance, let it be required to find the number of roots in the equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$.

The

The limiting equation is

$8x^7+28x^6-6x^5-50x^4+20x^3-15x^2-20x=10$

In this caſe the limiting equation is of a form capable of having as many roots as the propoſed equation, and it becomes now a queſtion to find the number of roots in the limiting equation. Inſtead of this, let the method adopted in pages 84, 85, 86, be applied to the propoſed equation.

$x^8+4x^7-x^6-10x^5+5x^4-5x^3-10x^2-10x=5.$

The unknown ſide may be divided into two parts $x^8+4x^7-x^6-10x^5$ and $5x^4-5x^3-10x^2-10x$, and the equation is of a form capable of having three roots. Now the part $5x^4-5x^3-10x^2-10x$ will, if $5x^4$ is greater than the other terms, make a number to be added to the firſt part, and if it is leſs, make a number to be taken from the firſt part. The ſame may be obſerved of the firſt part if x^8+4x^7 is greater or leſs than x^6+10x^5. Now x muſt evidently be greater than unity, and $5x^4-5x^3-10x^2-10x$ makes a number to be taken from the firſt part if x is not greater than 2. But when x is equal to 2, or ſome number leſs than 2, x^8+4x^7 is greater than x^6+10x^5. Conſequently ſome number leſs than 2 will make x^8+4x^7 equal to x^6+10x^5, and then every addition to x increaſes the part $x^8+4x^7-x^6-10x^5$. Now if the part $10x+10x^2+5x^3-5x^4$ decreaſes by the addition of x to x the whole unknown ſide muſt neceſſarily increaſe. The part $10x+10x^2+5x^3-5x^4$ firſt increaſes and then decreaſes, and the limit of its increaſe is when

$$10 = 20x^3 - 15x^2 - 20x$$
$$\text{or } ,5 = x^3 - ,75x^2 - x$$

that is x is not equal to 1,2, and consequently when x is greater than 1,2 the part $10x + 10x^3 + 5x^3 - 5x^4$ conſtantly diminiſhes till $10x + 10x^2 + 5x^3$ is equal to $5x^4$, and then by adding $\dot{x}$ to x as before, the part $5x^4 - 5x^3 - 10x^2 - 10x$ conſtantly increaſes. But when x is equal to 1,2 the unknown ſide of the given equation is no number at all, and conſequently when the unknown ſide by x being a greater number, is equal to 5 the part $x^8 + 4x^7 - x^6 - 10x^5$ is increaſed by the addition of $\dot{x}$ to x, and the part to be taken from it is either leſs than it was before, and conſequently the unknown ſide will be greater than 5, or inſtead of a number to be taken away from $x^8 + 4x^7 - x^6 - 10x^5$ the other part is a number to be added to it. Therefore the unknown ſide is always increaſed by adding $\dot{x}$ to x, or diminiſhed by taking away $\dot{x}$ from x and the equation can have only one root. This root is diſcovered by Baron Maſeres, in the Scriptores Logarithmici, Vol. III. p. 749, to be the only root, and to be very nearly equal to 1,618, and the equation is the higheſt numeral equation mentioned by Newton in his Arithmetica Univerſalis.

Let M be the number denoting the claſs, and N the number denoting an order in that claſs, then the number of orders in any claſs is equal to M, the number of forms in any claſs M is equal to $2^M - 1$, and the number of forms in any order N is equal to $M \cdot \frac{M-1}{2} \cdot \frac{M-2}{3} \cdot \frac{M-3}{4} \cdot \ldots\ldots \cdot \frac{M-N+1}{N}$.

Thus,

Thus, let M be equal to 3, then in the third claſs there are three orders. Let M be equal to 100, and in the 100th claſs there are 100 orders.

Let M be equal to 4, then in the fourth claſs the number of forms is equal to $2^4 - 1$ or $16 - 1$ or 15.

Let $M = 4$ and $N = 3$, then the number of forms in the fourth claſs, capable of having three roots, is equal to $4 \times \frac{3}{2} \times \frac{2}{3}$ or 4. That is, there are four forms being capable of having three roots.

Let $M = 4$ and $N = 2$, then the number of forms in the fourth claſs, capable of having two roots, is $4 \times \frac{3}{2}$ or 6. That is, there are ſix forms capable of having two roots.

If the unknown term, having the higheſt index in an equation, is to be taken away from the reſt, the number of the order to which it belongs is an even number, and conſequently the greateſt number of roots an equation of that form can have is an even number; but if the unknown term, having the higheſt index, is not to be taken away from the reſt, then the order to which it belongs is an odd number, and conſequently the greateſt number of roots an equation of that form can have is an odd number.

Let N be the number denoting the order to which an equation belongs, then the number of the changes of the marks is equal to N or $N - 1$. Hence, by ob-

obſerving the changes of the marks in any equation, the greateſt number of roots which an equation of that form is capable of having is aſcertained. Let the number of changes be P, then, if the term with the higheſt index is the firſt term, N is an odd number, otherwiſe N is an even number. If P then is an odd number, and N an even number, P is equal to N — 1 or N is equal to P + 1; but if both P and N are even numbers or odd numbers then P is equal to N.

Let $x^3 - 16x^2 + 77x = 98$.

x^3 being the firſt term, the order to which this equation belongs is denoted by an odd number, but there is an even number of changes, therefore N = P + 1 or 3; that is, this equation cannot have more than three roots, and it depends upon the coparts and known term, whether it has one, two, or three roots.

Let $x^8 + 4x^7 - x^6 - 10x^3 + 5x^4 - 5x^3 - 10x^2 - 10x = 5$:

x^8 being the firſt term, the equation belongs to an order whoſe index is an odd number, and the number of changes in the marks is odd, therefore the order to which this equation belongs is the third; this equation cannot have more than three roots. If the equation had been

$$x^8 - 4x^7 + x^6 - 10x^3 + 5x^4 - 5x^3 + 10x^2 - 10x = 5,$$

by counting the unknown terms it appears that it belongs to the eighth claſs, and conſequently cannot have more than eight roots: ſince x^8 is the firſt term, the number denoting the order is an odd number, and there being

ſeven,

ſeven, that is an odd number of changes in the marks, the equation belongs to the ſeventh order, and cannot conſequently have more than ſeven roots.

The inveſtigation of the properties of equations is endleſs; it is with them as with intelligent beings. There is no limit to the number of modes of each form. There is no limit to the number of forms. There is no limit to the number of orders. There is no limit to the number of claſſes. Each mode has its peculiar curve. The lives of men of the firſt talents have been employed upon a ſingle curve, and there are not names given to a hundred ſpecies of curves. By the claſs of intelligent beings next in rank above man, all theſe equations and all theſe curves are, perhaps, thoroughly underſtood, and the next claſs excells them as much as they do us. How great then muſt be that being to whom the thoughts of all theſe orders of beings are known at a moment's glance; and how inſignificant in the eye of reaſon are thoſe nations which lay down rules for thought and perſecute for opinions.

THE END.